

Toroidal maps : Schnyder woods, orthogonal surfaces and straight-line representations

Daniel Gonçalves, Benjamin Lévêque

February 7, 2012

Abstract

A Schnyder wood is an orientation and coloring of the edges of a planar map satisfying a simple local property. We propose a generalization of Schnyder woods to toroidal maps with application to graph drawing. We prove the existence of these Schnyder woods for toroidal triangulations. We show that Schnyder woods can be used to embed the universal cover of a toroidal map on an infinite and periodic orthogonal surface. Finally we use this embedding to obtain a straight-line flat torus representation of any toroidal map in a polynomial size grid.

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1 Introduction

A closed curve on a surface is *contractible* if it can be continuously transformed into a single point. Given a toroidal map, a *contractible loop* is an edge forming a contractible cycle. Two *homotopic multiple edges* are two edges with the same extremities such that their union forms a contractible cycle. We consider maps on the torus with no contractible loop and no homotopic multiple edges.

A general graph (i.e. not embedded on a surface) is *simple* if it contains no loop and no multiple edges. Since some loops and multiple edges are allowed the class we consider is larger than the class of simple toroidal graphs.

The torus is represented by a parallelogram in the plane whose opposite sides are pairwise identified. This representation is called the *flat torus*. The *universal cover* G^∞ of a toroidal map G is the infinite planar graph obtained by replicating a flat torus representation of G to tile the plane (the tiling is obtained by translating the flat torus along two vectors corresponding to the sides of the parallelogram). Note that a toroidal map G has no contractible loop and no homotopic multiple edges if and only if G^∞ is simple.

Given a general graph G , let n be the number of vertices and m the number of edges. Given a map embedded on a surface, let f be the number of faces. Euler's formula says that any connected map embedded in a surface of genus g satisfies $n - m + f = 2 - 2g$. Where the plane is the surface of genus 0, and the torus the surface of genus 1.

Schnyder woods were originally defined for planar triangulations by Schnyder [24]:

Definition 1 *Given a planar triangulation G , a Schnyder wood is an orientation and coloring of the edges of G with the colors 0, 1, 2 where each inner vertex v satisfies the Schnyder property, (see Figure 1 where each color is represented by a different type of arrow):*

- *Vertex v has out-degree one in each color.*
- *The edges $e_0(v)$, $e_1(v)$, $e_2(v)$ leaving v in colors 0, 1, 2, respectively, occur in counterclockwise order.*
- *Each edge entering v in color i enters v in the counterclockwise sector from $e_{i+1}(v)$ to $e_{i-1}(v)$ (where $i + 1$ and $i - 1$ are understood modulo 3).*

For higher genus triangulated surfaces, a generalization of Schnyder wood has been proposed by Castelli Aleardi et al. [3], with applications to encoding. Unfortunately, in this definition, the simplicity and the symmetry of the original Schnyder wood are lost. Here we propose an alternative generalization of Schnyder woods for toroidal maps, with application to graph drawings.

By Euler's formula, a planar triangulation satisfies $m = 3n - 6$. Thus there is not enough edges in the graph for all vertices to be of out-degree three. This explains why

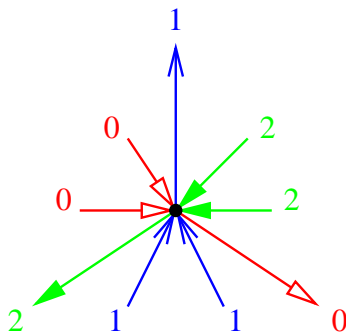


Figure 1: Schnyder property

just some vertices (inner ones) are required to verify the Schnyder property. For a toroidal triangulation, Euler's formula gives exactly $m = 3n$ so there is hope for a nice object satisfying the Schnyder property for every vertex. This paper shows that such an object exists. Here we do not restrict ourselves to triangulations and we directly define Schnyder woods in a more general framework.

Felsner [7, 8] (see also [19]) has generalized Schnyder woods to 3-connected planar maps by allowing edges to be oriented in one direction or in two opposite directions. We also allow edges to be oriented in two directions in our definition:

Definition 2 *Given a toroidal map G , a Schnyder wood of G is an orientation and coloring of the edges of G with the colors 0, 1, 2, where every edge e is oriented in one direction or in two opposite directions (each direction having a distinct color), satisfying the following (see example of Figure 2):*

- (T1) *Every vertex v satisfies the Schnyder property (see Definition 1)*
- (T2) *Every monochromatic cycle of color i intersects at least one monochromatic cycle of color $i - 1$ and at least one monochromatic cycle of color $i + 1$.*

In the case of toroidal triangulations, $m = 3n$ implies that there are too many edges to have bi-oriented edges. Thus, we can use this general definition of Schnyder wood for toroidal maps and keep in mind that when restricted to toroidal triangulations all edges are oriented in one direction only.

We prove the existence of Schnyder woods for any toroidal triangulation and conjecture that it exists for any essentially 3-connected toroidal map.

Theorem 1 *A toroidal triangulation admits a Schnyder wood.*

In our definition of Schnyder woods, two properties are required : a local one (T1) and a global one (T2). This second property is required to use Schnyder woods to embed toroidal maps on orthogonal surfaces like it has been done in the plane by Miller [19] (see also [8]).

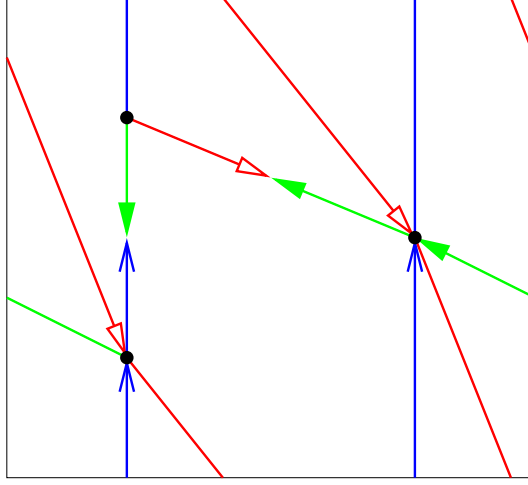


Figure 2: Example of a Schnyder wood of a toroidal map.

Theorem 2 *The universal cover of a toroidal triangulation admits a geodesic embedding on an infinite and periodic orthogonal surface.*

In the embeddings obtained by Theorem 2, vertices are not coplanar but one can project the vertices on a plane to obtain a periodic straight line representation of the universal cover. The restriction of such a representation to a flat torus is a *straight line flat torus representation*. The problem to find a straight line flat torus representation of a toroidal map was previously solved on exponential size grids [20]. There are several works to represent a toroidal map inside a parallelogram in a polynomial size grid [4, 5], but in these representations the opposite sides of the parallelogram do not perfectly match. Thus to our knowledge, the work presented here gives the first straight line flat torus representation in a polynomial size grid.

For planar maps, straight line representations in polynomial size grids also have the property that edges have polynomial lengths. For toroidal maps, this is not necessarily the case as an edge can turn several times around the flat representation. In our representations the edges have polynomial lengths.

Theorem 3 *A toroidal map admits a straight line flat torus representation in a polynomial size grid with edges having polynomial length.*

The polynomial bounds obtained in Theorem 3 depends on the “simplicity” of the toroidal map. In general we prove that the grid has size bounded by $\mathcal{O}(n^4) \times \mathcal{O}(n^4)$ and we can improve this bound to $\mathcal{O}(n^2) \times \mathcal{O}(n^2)$ if the graph is simple. The length of the edges are bounded by $\mathcal{O}(n^3)$ in general and by $\mathcal{O}(n^2)$ if the graph is simple.

2 Generalization of the planar case

Felsner [7, 8] has generalized Schnyder woods by allowing edges to be oriented in one direction or in two opposite directions. The formal definition is the following:

Definition 3 *Given a planar map G . Let x_0, x_1, x_2 be three distinct vertices occurring in counterclockwise order on the outer face of G . The suspension G^σ is obtained by attaching a half-edge that reaches into the outer face to each of these special vertices. A Schnyder wood rooted at x_0, x_1, x_2 is an orientation and coloring of the edges of G^σ with the colors 0, 1, 2, where every edge e is oriented in one direction or in two opposite directions (each direction having a distinct color), satisfying the following (see example of Figure 3):*

- (P1) *Every vertex v satisfies the Schnyder property and the half-edge at x_i is directed outwards and colored i*
- (P2) *There is no monochromatic cycle.*

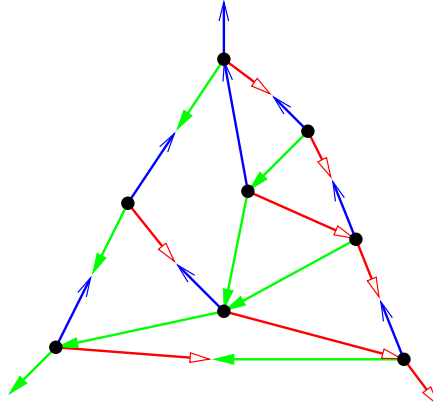


Figure 3: Example of a Schnyder wood of a planar map.

In the definition given by Felsner [8], property (P2) is in fact: “There is no interior face the boundary of which is a monochromatic cycle”, but the two are equivalent by results of [7, 8].

With our definition of Schnyder wood for toroidal map, the goal is to generalize the definition of Felsner. In the torus, property (P1) can be simplified as every vertex plays the same role: there is no special outer vertices with a half-edge reaching into the outer face. This explain property (T1) in our definition. Then if one asks that every vertex satisfies the Schnyder property, there is necessarily monochromatic cycles and (P2) is not satisfied. This explain why (P2) has been modified in our definition.

Let G be a toroidal map given with a Schnyder wood. Let G_i be the directed graph induced by the edges of color i . This definition includes edges that are half-colored i ,

and in this case, only the edges gets the direction corresponding to color i . Each graph G_i has exactly n edges, so it does not induce a rooted tree (contrarily to planar Schnyder woods) and the term “wood” has to be handle with care here. Note also that G_i is not necessarily connected (for example in the map of Figure 4, every Schnyder wood has one color which corresponding subgraph is not connected). But each components of G_i has exactly one outgoing arc for each of its vertices, thus each connected component of G_i has exactly one directed cycle that is a *monochromatic cycle* of color i , or *i-cycle* for short. Note that monochromatic cycles can contain edges oriented in two directions with different colors, but the orientation of a *i-cycle* is the orientation given by color i .

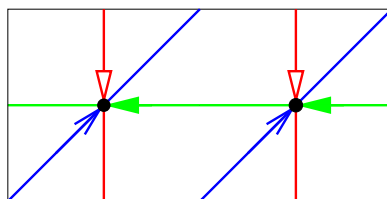


Figure 4: A toroidal map where every Schnyder wood has one color which corresponding subgraph is not connected

A closed curve on a surface is *contractible* if it can be continuously transformed into a single point. Every closed curve in the plane is contractible but this is not true on general surfaces. One can replace (P2) by “there is no contractible monochromatic cycles” but this is not enough to suit our needs. Our goal is to use Schnyder woods to embed universal cover of toroidal maps on orthogonal surfaces like it has been done in the plane by Miller [19] (see also [8]). The difference being that that our surface is infinite and periodic. In such a representation the three colors 0, 1, 2 corresponds to the three directions of the space. Thus the monochromatic cycles with different colors have to intersect each other in a particular way. This explains why property (T2) is required. Figure 5 gives an example of an orientation and coloring of the edges of a toroidal triangulation satisfying (T1) but not (T2) as there is no pair of intersecting monochromatic cycles.

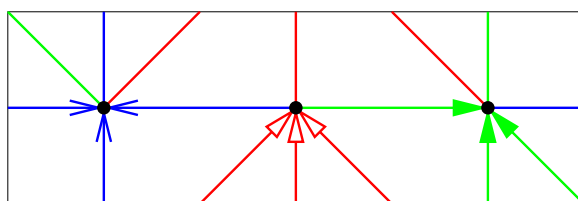


Figure 5: An orientation and coloring of the edges of a toroidal triangulation satisfying (T1) but not (T2) as there is no pair of intersecting monochromatic cycles.

Let G be a toroidal map given with a Schnyder wood. The graph G_i^{-1} is the graph obtained from G_i by reversing all its edges. The graph $G_i \cup G_{i-1}^{-1} \cup G_{i+1}^{-1}$ is obtained

from the graph G by orienting edges in one or two direction depending on whether this orientation is present in G_i , G_{i-1}^{-1} or G_{i+1}^{-1} . The following Lemma shows that our property (T2) in fact implies that there is no contractible monochromatic cycles.

Lemma 1 *The graph $G_i \cup G_{i-1}^{-1} \cup G_{i+1}^{-1}$ contains no contractible directed cycle.*

Proof. Suppose there is a contractible directed cycle in $G_i \cup G_{i-1}^{-1} \cup G_{i+1}^{-1}$. Let C be such a cycle containing the minimum number of faces in the closed disk D bounded by C . Suppose by symmetry that C turns around D clockwise. Then, by (T1), there is no edge of color $i-1$ leaving closed disk D . So there is a $(i-1)$ -cycle in D and this cycle is C by minimality of C . Then, by (T1), there is no edge of color i leaving D , so similarly, the cycle C is a i -cycle. Then, by (T1), all the edges of color $i+1$ incident to C have to leave D . Thus there is no $(i+1)$ -cycle intersecting C , a contradiction to property (T2). \square

Lemma 1 is useful to prove that our definition of Schnyder wood for toroidal maps generalize the planar case.

Let G be a planar map and x_0, x_1, x_2 be three distinct vertices occurring in counter-clockwise order on the outer face of G . One can transform G^σ into the following toroidal map G^+ (see Figure 6): Add a vertex v in the outer face of G . Add three non-parallel and non-contractible loops on v . Connect the three half edges leaving x_i to v such that there is no two such edge entering v consecutively. Then we have the following.

Theorem 4 *The Schnyder woods of a planar map G rooted at x_0, x_1, x_2 are in bijection with the Schnyder woods of the toroidal map G^+ .*

Proof. (\implies) Given a Schnyder wood of the planar graph G , rooted at x_0, x_1, x_2 . Orient and color the graph G^+ as in the example of Figure 6, i.e. the edges of the original graph G have the same color and orientation as in G^σ , the edge from x_i to v is colored i and leaving x_i , the three loops around v are colored and oriented appropriately so that v satisfies the Schnyder property. Then it is clear that all the vertices of G^+ satisfy (T1). By (P2), we know that G^σ has no monochromatic cycles. All the edges between G and v are leaving G , so there is no monochromatic cycle of G^+ involving vertices of G . Thus the only monochromatic cycles of G^+ are the three loops around v and they satisfy (T2).

(\impliedby) Given a Schnyder wood of G^+ , the restriction of the orientation and coloring to G and the three edges leaving v gives a Schnyder wood of G^σ . The three loops around v are three monochromatic cycles and thus have different colors. Thus the three edges between G and v are entering v with three different colors. The three loops around v have to leave v in counterclockwise order 0, 1, 2 and we can assume by maybe permuting the colors that the edge leaving x_i is colored i . Clearly all the vertices of G^σ satisfies (P1). By Lemma 1, there is no contractible monochromatic cycles in G^+ , so G^σ satisfies (P2). \square

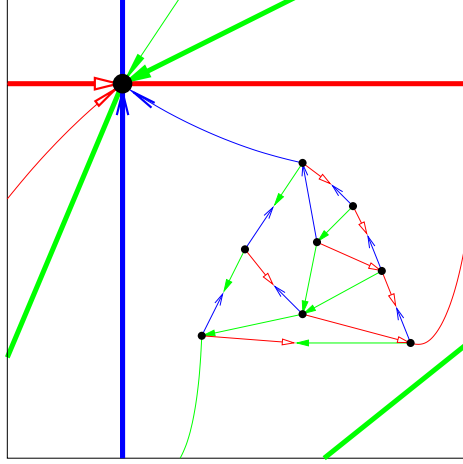


Figure 6: The toroidal Schnyder wood corresponding to the planar Schnyder wood of Figure 3

3 Properties of Schnyder woods

Two non contractible closed curves are *homotopic* if one can be continuously transformed into the other. The following are general useful lemmas on the torus.

Lemma 2 *Let C_1, C_2 be two non contractible closed curve on the Torus. If C_1, C_2 are not homotopic, then their intersection is non empty.*

Lemma 3 *Let C_1, C_2, C_3 be three non contractible closed curve on the Torus. If C_1, C_2 are homotopic and C_2, C_3 are homotopic, then C_1, C_3 are homotopic.*

Lemma 4 *Let C_1, C_2, C_3 be three non contractible closed curve on the Torus. If C_1, C_2 are homotopic and C_1, C_3 are not homotopic. Then C_2, C_3 are not homotopic and thus their intersection is non empty.*

Two non contractible oriented closed curves on the torus are *fully-homotopic* if one can be continuously transformed into the other by preserving the orientation. We say that two monochromatic directed cycles C_i, C_j of different colors are *reversal* if one is obtained from the other by reversing all the edges ($C_i = C_j^{-1}$). We say that two monochromatic cycle are *crossing* if they intersects but are not reversal. We define the *right side* of a i -cycle C_i , as the right side while “walking” along the directed cycle by following the orientation given by the edges colored i .

Let G be a toroidal map given with a Schnyder wood.

Lemma 5 *All i -cycles are non contractible, non intersecting and fully-homotopic.*

Proof. By Lemma 1, all i -cycles are non contractible. If there exists two such distinct i -cycles that are intersecting. Then there is vertex that has two outgoing edge of color i , a contradiction to (T1). So the i -cycles are non intersecting. Then, by Lemma 2, they are homotopic.

Suppose that there exists two i -cycles C_i, C'_i that are not fully-homotopic. By the first part of the proof, cycles C_i, C'_i are non contractible, non intersecting and homotopic. Let R be the region between C_i and C'_i situated on the right of C_i . Suppose by symmetry that C_i^{-1} is not a $(i+1)$ -cycle. By (T2), there exists a cycle C_{i+1} intersecting C_i and thus C_{i+1} is crossing C_i . By Property (T1), C_{i+1} is entering C_i from its right side and so it is leaving the region R when it crosses C_i . To enter the region R , the cycle C_{i+1} has to enter C_i or C'_i from their left side, a contradiction to property (T1). \square

The *length* of a path is its number of edges. A loop is a path of length 1. There always exists a path of length 0 between a vertex and itself.

We say that an edge e , incident to w , is in the sector $[e_1, e_2]$ of w , for e_1 and e_2 two edges incident to w , if e is in the counterclockwise sector of w between e_1 and e_2 , including the edge e_1 and e_2 . The sector $]e_1, e_2]$, $[e_1, e_2[$, and $]e_1, e_2[$ are defined analogously by excluding the corresponding edges from the sectors. We say that a directed cycle C (or path) enters (resp. leaves) w in a given sector, if C has an edge entering w (resp. leaving w) in this sector.

Lemma 6 *Let u, v be two vertices of G (that are not necessarily distinct). Suppose that there is a directed path Q_{i-1} of length ≥ 1 and of color $i-1$ from u to v , a directed path Q_{i+1} of length ≥ 1 and of color $i+1$ from u to v , a directed path Q_i of length 0 or 1 and color i from u to v (Q_i is of length 0 if $u = v$). Consider the two directed cycles $C_{i-1} = Q_{i-1} \cup Q_i^{-1}$ and $C_{i+1} = Q_{i+1} \cup Q_i^{-1}$. Then either C_{i-1} and C_{i+1} are not homotopic, or $u = v$ and $C_{i-1} = C_{i+1}^{-1}$.*

Proof. By Lemma 1, the cycles C_{i-1} and C_{i+1} are not contractible. Suppose that C_{i-1} and C_{i+1} are homotopic, and $u \neq v$ or $C_{i-1} \neq C_{i+1}^{-1}$. Let fix the positive side of C_{i-1} has the side containing the counterclockwise sector $[e_{i+1}(u), e_{i-1}(u)]$. If $u \neq v$, then the path Q_{i+1} is leaving C_{i-1} at u . If $u = v$, since $C_{i-1} \neq C_{i+1}^{-1}$, then again Q_{i+1} is leaving C_{i-1} (maybe not at u). In both cases, by (T1), Q_{i+1} is leaving C_{i-1} on its positive side. Since C_{i-1} and C_{i+1} are homotopic, the path Q_{i+1} is entering C_{i-1} at least once from the positive side. This is in contradiction with (T1). \square

Lemma 7 *Let w, x, y be three vertices such that $e_{i-1}(x)$ and $e_{i+1}(y)$ are entering w . Suppose that there is a directed path Q_{i-1} of color $i-1$ from y to w , entering w in the sector $[e_i(w), e_{i-1}(x)]$, and a directed path Q_{i+1} of color $i+1$ from x to w , entering w in the sector $[e_{i+1}(y), e_i(w)]$. Consider the two directed cycles $C_{i-1} = Q_{i-1} \cup \{e_{i+1}(y)\}^{-1}$ and $C_{i+1} = Q_{i+1} \cup \{e_{i-1}(x)\}^{-1}$. Then either C_{i-1} and C_{i+1} are not homotopic or $C_{i-1} = C_{i+1}^{-1}$.*

Proof. By Lemma 1, the cycles C_{i-1} and C_{i+1} are not contractible. Suppose that C_{i-1} and C_{i+1} are homotopic and that $C_{i-1} \neq C_{i+1}^{-1}$. Let fix the positive side of C_{i-1} has the side containing the sector $[e_{i+1}(y), e_{i-1}(y)]$. Since $C_{i-1} \neq C_{i+1}^{-1}$, the cycle C_{i+1} is leaving C_{i-1} . By (T1) it is leaving C_{i-1} on the positive side. Since C_{i-1} and C_{i+1} are homotopic, the path Q_{i+1} is entering C_{i-1} at least once from the positive side. This is in contradiction with (T1). \square

Lemma 8 *If two monochromatic cycles are crossing then they are of different colors and they are not homotopic.*

Proof. By Lemma 5, the two crossing monochromatic cycles are not of the same color. By Lemma 6, with $u = v$ a vertex of the intersection, the two cycles are not homotopic \square

Let \mathcal{C}_i be the set of i -cycles of G . Let $(\mathcal{C}_i)^{-1}$ denote the set of cycles obtained by reversing all cycles \mathcal{C}_i . By Lemma 5, the cycles of \mathcal{C}_i are non contractible, non intersecting and fully-homotopic. So we can order them as follow $\mathcal{C}_i = \{C_i^0, \dots, C_i^{k_i-1}\}$, $k_i \geq 1$, such that, for $0 \leq j \leq k_i - 1$, there is no i -cycle in the region $R(C_i^j, C_i^{j+1})$ between C_i^j and C_i^{j+1} containing the right side of C_i^j (superscript understood modulo k_i).

Schnyder woods are only of two different types (see figure 7 and Theorem 5).

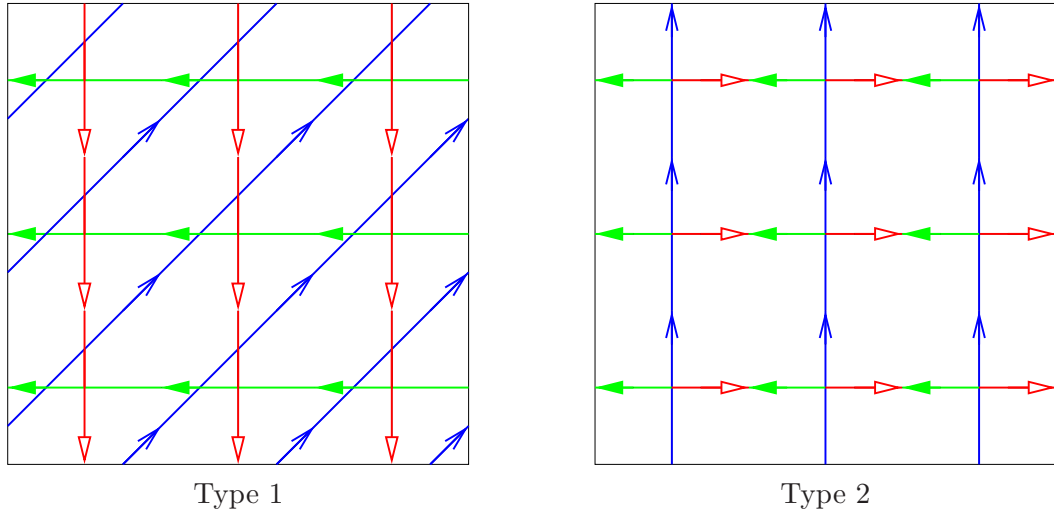


Figure 7: The two types of Schnyder woods on toroidal maps

Theorem 5 *Let G be a toroidal map given with a Schnyder wood, then either:*

- *For every pair of two monochromatic cycles C_i, C_j of different colors i, j , the two cycles C_i and C_j are not homotopic and thus intersect (We say the Schnyder wood is of Type 1).*

or

- There exists a color i such that $\mathcal{C}_{i-1} = (\mathcal{C}_{i+1})^{-1}$ and for any pair of monochromatic cycles C_i, C_j of colors i, j , with $j \neq i$, the two cycles C_i and C_j are not homotopic and thus intersect (We say the Schnyder wood is of Type 2, or Type 2.i if we want to specify the color i).

Moreover, if G is a toroidal triangulation, then there is no edges oriented in two directions and the Schnyder wood is of Type 1.

Proof. Suppose that there exists a $(i-1)$ -cycle C_{i-1} and a $(i+1)$ -cycle C_{i+1} that are homotopic. We prove that the Schnyder wood is of Type 2.i. We first prove that $\mathcal{C}_{i-1} = (\mathcal{C}_{i+1})^{-1}$. Let C'_{i-1} be any $(i-1)$ -cycle. By (T2), C'_{i-1} intersects a $(i+1)$ -cycle C'_{i+1} . By Lemma 5, C'_{i-1} (resp. C'_{i+1}) is homotopic to C_{i-1} (resp. C_{i+1}). So, by Lemma 3, C'_{i-1} and C'_{i+1} are homotopic. By Lemma 8, C'_{i-1} and C'_{i+1} are reversal. Thus $\mathcal{C}_{i-1} \subseteq (\mathcal{C}_{i+1})^{-1}$ and so by symmetry $\mathcal{C}_{i-1} = (\mathcal{C}_{i+1})^{-1}$. Now we prove that for any pair of monochromatic cycles C'_i, C'_j of colors i, j , with $j \neq i$, the two cycles C'_i and C'_j are not homotopic. By (T2), C'_j intersects a i -cycle C_i . Since $\mathcal{C}_{i-1} = (\mathcal{C}_{i+1})^{-1}$, cycle C'_j is bi-oriented in color $i-1$ and $i+1$, thus we cannot have $C'_j = C_i^{-1}$. So C'_j and C_i are crossing and by Lemma 8, they are not homotopic. By Lemma 5, C'_i and C_i are homotopic. Thus, by Lemma 4, C'_j and C'_i are not homotopic. Thus the Schnyder wood is of Type 2.i.

If there is no two monochromatic cycles of different colors that are homotopic, then the Schnyder wood is of Type 1.

For toroidal triangulation, $m = 3n$ by Euler's formula, so there is no edges oriented in two directions, so only Type 1 is possible. \square

Note that in a Schnyder wood of Type 1, we may have edges that are in two monochromatic cycles of different colors (see Figure 2).

The next lemma shows that we can relax a bit (T2) for toroidal triangulations. In fact the following property (T2') is enough to have a Schnyder wood:

- (T2') For each pair i, j of different colors, there exists a monochromatic cycle of color i intersecting a monochromatic cycle of color j .

Lemma 9 *Given a toroidal triangulation G , and an orientation and coloring of the edges of G with the colors 0, 1, 2, where every edge e is oriented in one direction only, satisfying property (T1) and (T2'). Then (T2) is satisfied and so the orientation and coloring is a Schnyder wood.*

Proof. We have to prove that several properties of above lemmas are still true within the new hypothesis. Note that $m = 3n$ so there is no edges oriented in two directions,

(1) *All i -cycles are non contractible, non intersecting and homotopic.*

Suppose there is a contractible monochromatic cycle. Let C be such a cycle containing the minimum number of faces in the closed disk D bounded by C . Suppose by symmetry that C turns around D clockwise. Let i be the color of C . Then, by (T1), there is no edge of color $i - 1$ leaving the closed disk D . So there is a $(i - 1)$ -cycle in D and this cycle is C by minimality of C . So C contains edges oriented in two directions, a contradiction. If there exists two distinct i -cycles that are intersecting. Then there is vertex that has two outgoing edge of color i , a contradiction to (T1). So the i -cycles are non intersecting. Then, by Lemma 2, they are homotopic. This proves claim (1). \diamond

(2) *If two monochromatic cycles are intersecting then they are not homotopic.*

Suppose by contradiction that there exists C, C' two distinct directed monochromatic cycles that are homotopic and intersecting. By Claim (1), they are not contractible and of different color. Suppose C is a $(i - 1)$ -cycle and C' and $(i + 1)$ -cycle. By (T1), C' is leaving C on its right side. Since C, C' are homotopic, the cycle C' is entering C at least once from its right side, a contradiction with (T1). This proves claim (2). \diamond

We now prove that (T2) is satisfied. Let C_i be any i -cycle of color i . We have to prove that C_i intersects at least one $(i - 1)$ -cycle and at least one $(i + 1)$ -cycle. Let j be either $i - 1$ or $i + 1$. By (T2'), there exists a i -cycle C'_i intersecting a j -cycle C'_j of color j . The two cycles C'_i, C'_j are not reversal as there is no edge oriented in two directions, thus they are crossing. By Claim (2), C'_i and C'_j are not homotopic. By Claim (1), C_i and C'_i are homotopic. Thus by Lemma 4, C_i and C'_j are intersecting. \square

Lemma 9 is not true for general toroidal maps. Figure 8 gives an example of an orientation and coloring of the edges of a toroidal map satisfying (T1) and (T2') but that is not a Schnyder wood. There is a monochromatic cycle of color 1 that is not intersecting any monochromatic cycle of color 2.

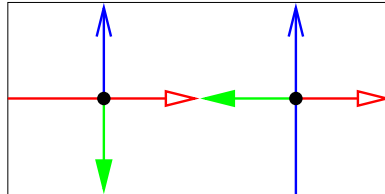


Figure 8: An orientation and coloring of the edges of toroidal map satisfying (T1) and (T2') but that is not a Schnyder wood.

We do not know if the set of Schnyder woods of a given toroidal map has a kind of lattice structure like in the planar case. The following remark suggests that the usual ideas to prove that Schnyder woods are in bijection with α -orientation and thus have a lattice structure (see [9]) do not work similarly for the Torus.

De Fraysseix et al. [14] proved that Schnyder woods of a planar triangulation are in one-to-one correspondence with orientation of the edges of the graph where each inner vertex has out-degree three. It is possible to retrieve the coloring of the edges of a Schnyder wood from the orientation. The situation is different for toroidal triangulations. There exists orientations of toroidal triangulations where each vertex has out-degree three but there is no corresponding Schnyder wood. For example, if one consider a toroidal triangulation with just one vertex, the orientations of edges that satisfies (T1) are the orientations where there is no three consecutive edges leaving the vertex (see Figure 9). Note that there are also such examples that are simple triangulations like some orientations of K_7 .

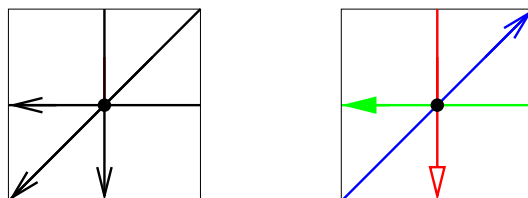


Figure 9: Two different orientations of a toroidal triangulation. Only the second one corresponds to a Schnyder wood.

4 Schnyder woods in the universal cover

Let G be a toroidal map given with a Schnyder wood. Consider the orientation and coloring of the edges of G^∞ that corresponds to the Schnyder wood of G .

Lemma 10 *The orientation and coloring of the edges of G^∞ satisfies the following:*

(U1) *Every vertex of G^∞ verifies the Schnyder property*

(U2) *There is no monochromatic cycle in G^∞ .*

Proof. Clearly, (U1) is satisfied. Now we prove (U2). Suppose by contradiction that there is a monochromatic cycle U of color i in G^∞ . A cycle of G^∞ do not necessarily corresponds to a cycle in G as it may self intersect in G . Let C be the closed curve of G corresponding to edges of U . If C self intersects, then there is a vertex of G with two edges leaving v in color i , a contradiction to (T1). So C is a monochromatic cycle of G . Since C corresponds to a cycle of G^∞ , it is a contractible cycle of G , a contradiction to Lemma 1. \square

One can remark that properties (U1) and (U2) are the same as in the definition of Schnyder wood for 3-connected planar graphs (properties (P1) and (P2)). Note that properties (U1) and (U2) are not enough to have a Schnyder wood of G . For example the graph G^∞ obtained by replicating the graph G of Figure 5 satisfies (U1) and (U2) whereas the orientation and coloring of G is not a Schnyder wood as (T2) is not satisfied.

Recall that the notation $\mathcal{C}_i = \{C_i^0, \dots, C_i^{k_i-1}\}$ denotes the set of i -cycles of G such that there is no i -cycle in the region $R(C_i^j, C_i^{j+1})$. A directed monochromatic cycles C_i^j corresponds to a family of infinite directed monochromatic paths of G^∞ (infinite in both directions of the path). This family is denoted \mathcal{L}_i^j . Each element of \mathcal{L}_i^j is called a *monochromatic line* of color i , or i -line for short. By Lemma 5, all i -lines are non intersecting and oriented in the same direction. Given any two i -lines L, L' , the unbounded region between L and L' is noted $R(L, L')$. We say that two i -lines L, L' are *consecutive* if there is no i -lines contained in $R(L, L')$.

Let v be a vertex of G^∞ . For each color i , vertex v is the starting vertex of a unique infinite directed monochromatic path of color i , denoted $P_i(v)$. Indeed this is a path since there is no monochromatic cycle (U2), and it is infinite (in one direction of the path only) because every reached vertex has exactly one edge leaving in color i (U1). Let $L_i(v)$ be the i -line intersecting $P_i(v)$.

Lemma 11 *The graph $G_i^\infty \cup (G_{i-1}^\infty)^{-1} \cup (G_{i+1}^\infty)^{-1}$ contains no contractible directed cycle.*

Proof. Suppose there is a contractible directed cycle C in $G_i^\infty \cup (G_{i-1}^\infty)^{-1} \cup (G_{i+1}^\infty)^{-1}$. Let D be the closed disk bounded by C . Suppose by symmetry that C turns around D clockwise. Then, by (U1), there is no edge of color $i-1$ leaving the closed disk D . So there is a $(i-1)$ -cycle in D , a contradiction to (U2). \square

Lemma 12 *For every vertex v and color i , the two paths $P_{i-1}(v)$ and $P_{i+1}(v)$ have v as only common vertex.*

Proof. If $P_{i-1}(v)$ and $P_{i+1}(v)$ intersect on two vertices, then G^∞ contains a cycle, contradicting Lemma 11. \square

By Lemma 12, for every vertex v , the three paths $P_0(v)$, $P_1(v)$, $P_2(v)$ divide G^∞ into three unbounded regions $R_0(v)$, $R_1(v)$ and $R_2(v)$, where $R_i(v)$ denotes the region delimited by the two paths $P_{i-1}(v)$ and $P_{i+1}(v)$. Let $R_i^\circ(v) = R_i(v) \setminus (P_{i-1}(v) \cup P_{i+1}(v))$.

Lemma 13 *For all distinct vertices u, v , we have:*

- (i) *If $u \in R_i(v)$, then $R_i(u) \subseteq R_i(v)$.*
- (ii) *If $u \in R_i^\circ(v)$, then $R_i(u) \subsetneq R_i(v)$.*
- (iii) *There exists i and j with $R_i(u) \subsetneq R_i(v)$ and $R_j(v) \subsetneq R_j(u)$.*

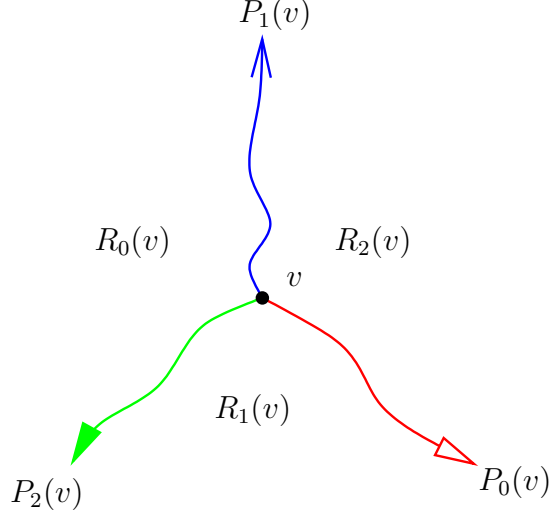


Figure 10: Regions corresponding to a vertex

Proof. (i) Suppose by symmetry that the Schnyder wood is not of Type 2.(i+1). Then in G , i -cycles are not homotopic to $(i-1)$ -cycles. Thus in G^∞ , every i -line crosses every $(i-1)$ -line. Moreover a i -line crosses a $(i-1)$ -line exactly once and from its right side to its left side by (U1). Vertex v is between two consecutive monochromatic $(i-1)$ -lines L_{i-1}, L'_{i-1} , with L'_{i-1} situated on the right of L_{i-1} . Let R be the region situated on the right of L_{i-1} , so $v \in R$.

(3) For any vertex w of R , the path $P_i(w)$ leaves the region R .

The i -line $L_i(w)$ has to cross L_{i-1} exactly once and from right to left, thus $P_i(w)$ leaves the region R . This proves claim (3). \diamond

The path $P_{i+1}(v)$ cannot leave the region R as this would contradict (U1). Thus by Claim 3 for $w = v$, we have $R_i(v) \subseteq R$ and so $u \in R$. Moreover the paths $P_{i-1}(u)$ and $P_{i+1}(u)$ cannot leave region $R_i(v)$ as the vertex where one of them leaves would not satisfy (U1). Thus by Claim 3 for $w = u$, the path $P_i(u)$ leaves the region $R_i(v)$ and so $R_i(u) \subseteq R_i(v)$.

(ii) By (i), $R_i(u) \subseteq R_i(v)$, so the paths $P_{i-1}(u)$ and $P_{i+1}(u)$ are contained in $R_i(v)$. Then none of them can contain v as this would contradict (U1) on v . So all the faces of $R_i(v)$ incident to v are not in $R_i(u)$ (and there is at least one such face).

(iii) By symmetry, we prove that there exists i with $R_i(u) \subsetneq R_i(v)$. If $u \in R_i^\circ(v)$ for some color i , then $R_i(u) \subsetneq R_i(v)$ by (ii). Suppose now that $u \in P_i(v)$ for some i . By Lemma 12, at least one of the two paths $P_{i-1}(u)$ and $P_{i+1}(u)$ does not contain v . Suppose by symmetry that $P_{i-1}(u)$ does not contain v . As $u \in P_i(v) \subseteq R_{i+1}(v)$, we

have $R_{i+1}(u) \subseteq R_{i+1}(v)$ by (i), and as none of $P_{i-1}(u)$ and $P_i(u)$ contains v , we have $R_{i+1}(u) \subsetneq R_{i+1}(v)$. \square

Lemma 14 *For any vertex v , the two monochromatic lines $L_{i-1}(v)$ and $L_{i+1}(v)$ intersect. Moreover, if the Schnyder wood is of Type 2.i, then $L_{i+1}(v) = (L_{i-1}(v))^{-1}$ and v is situated on the right of $L_{i+1}(v)$.*

Proof. Let j, j' be such that $L_{i-1}(v) \in \mathcal{L}_{i-1}^j$ and $L_{i+1}(v) \in \mathcal{L}_{i+1}^{j'}$. If the Schnyder wood is of Type 1 or Type 2.j with $j \neq i$, then the two cycles C_{i-1}^j and $C_{i+1}^{j'}$ are not homotopic, and so the two lines $L_{i-1}(v)$ and $L_{i+1}(v)$ intersect.

If the Schnyder wood is of Type 2.i, we consider the case where $v \in L_{i-1}(v)$, and the case where v does not belong to $L_{i-1}(v)$ nor $L_{i+1}(v)$. Then v lies between two consecutive $(i+1)$ -lines (which are also $(i-1)$ -lines). Let us denote those two lines L_{i+1} and L'_{i+1} , such that L'_{i+1} is situated on the right of L_{i+1} and $v \notin L'_{i+1}$. By property (T1), $P_{i+1}(v)$ and $P_{i-1}(v)$ cannot reach L'_{i+1} . Thus $L_{i+1} = L_{i+1}(v) = (L_{i-1}(v))^{-1}$. \square

5 Essentially 3-connected toroidal maps

A planar map G is *internally 3-connected* if there exists three vertices on the outer face such that the graph obtained from G by adding a vertex adjacent to the three vertices is 3-connected. Miller [19] proved that a planar map admits a Schnyder wood if and only if it is internally 3-connected (see also [7]).

We need to define the generalization of this class for the torus. Extending the notion of *essentially 2-connectedness* defined in [21], we say that a toroidal map G is *essentially 3-connected* if its universal cover is 3-connected. The following Lemma shows that the notion of essentially 3-connected is the natural generalization of internally 3-connected. (Recall that G^+ is the graph obtained by the operation of Figure 6.)

Theorem 6 *A planar map G is internally 3-connected if and only if there exists three vertices on the outer face of G such that G^+ is essentially 3-connected.*

Proof. (\implies) Let G be an internally 3-connected planar map. By definition, there exists three vertices x_0, x_1, x_2 on the outer face such that the graph G' obtained from G by adding a vertex adjacent to these three vertices is 3-connected. Let G'' be the graph obtained from G by adding three vertices y_0, y_1, y_2 that form a triangle and by adding the three edges $x_i y_i$. It is not difficult to check that G'' is 3-connected. Since G^∞ can be obtained from the (infinite) triangular grid, which is 3-connected, by gluing copies of G'' along triangles, G^∞ is clearly 3-connected. Thus G^+ is essentially 3-connected.

(\impliedby) Suppose there exists three vertices on the outer face of G such that G^+ is essentially 3-connected, i.e. G^∞ is 3-connected. A copy of G is contained in a triangle

$y_0y_1y_2$ of G^∞ . Let G'' be the subgraph of G^∞ induced by this copy plus the triangle, and let x_i be the unique neighbor of y_i in the copy of G . Since G'' is connected to the rest of G^∞ by a triangle, G'' is also 3-connected. Let us now prove that this implies that G is internally 3-connected for x_0, x_1 and x_2 . This is equivalent to say that the graph G' , obtained by adding a vertex z connected to x_0, x_1 and x_2 , is 3-connected. If G' had a (≤ 2) -separator $\{a, b\}$ or $\{a, z\}$, with a and $b \in V(G') \setminus \{z\}$, then $\{a, b\}$ or $\{a, y_i\}$, for some $i \in [0, 2]$, would be a (≤ 2) -separator of G'' . This would contradict the 3-connectedness of G'' . So G is internally 3-connected. \square

Lemma 15 *If a toroidal map G admits a Schnyder wood, then G is essentially 3-connected.*

Proof. Let u, v, x, y be any four distinct vertices of G^∞ . Let us prove that there exists a path between u and v in $G^\infty \setminus \{x, y\}$. Suppose by symmetry, that the Schnyder wood is of Type 1 or Type 2.1. Then the monochromatic lines of color 0 and 2 form a kind of grid, i.e. the 0-lines intersect all the 2-lines. Let L_0, L'_0 be 0-lines and L_2, L'_2 be 2-lines, such that u, v, x, y are all in the interior of the bounded region $R(L_0, L'_0) \cap R(L_2, L'_2)$.

By Lemma 12, the three paths $P_i(v)$, for $0 \leq i \leq 2$, are disjoint except on v . Thus there exists i , such that $P_i(v) \cap \{x, y\} = \emptyset$. Similarly there exists j , such that $P_j(u) \cap \{x, y\} = \emptyset$. The two paths $P_i(v)$ and $P_j(u)$ are infinite, so they intersect the boundary of $R(L_0, L'_0) \cap R(L_2, L'_2)$. Thus $P_i(v) \cup P_j(u) \cup L_0 \cup L'_0 \cup L_2 \cup L'_2$ contains a path from u to v in $G^\infty \setminus \{x, y\}$. \square

Note that if (T2) is not required in the definition of Schnyder wood, then Lemma 15 is false. Figure 11 gives an example of an orientation and coloring of the edges of a toroidal map satisfying (T1) but not (T2) (there is a 0-cycle not intersecting any 2-cycle), and that is not essentially 3-connected (indeed G^∞ is not connected).

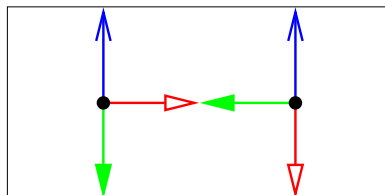


Figure 11: An orientation and coloring of the edges of a toroidal map satisfying (T1) but that is not essentially 3-connected.

6 Duality of Schnyder woods

Given a planar map G , and x_0, x_1, x_2 three distinct vertices occurring in counterclockwise order on the outer face of G . A *Schnyder angle labeling* [7] of G with respect to x_0, x_1, x_2 is a labeling of the angles of G^σ satisfying the following:

- (L1) The label of the angles at each vertex form, in counterclockwise order, nonempty intervals of 0's, 1's and 2's. The two angles at the half-edge at x_i have labels $i + 1$ and $i - 1$
- (L2) The label of the angles at each inner face form, in counterclockwise order, nonempty intervals of 0's, 1's and 2's. At the outer-face the same is true in clockwise order.

Felsner [8] proved that, for planar maps, Schnyder woods are in bijection with Schnyder angle labellings. In the toroidal case, we do not see a simple definition of Schnyder angle labeling that would be equivalent to our definition of Schnyder woods. This due to the fact that contrarily to (P2) that is local and can be verified just by considering faces, (T2) is global. Nevertheless we have one implication.

The *angle labeling corresponding to* a Schnyder wood of a toroidal map G is a labeling of the angles of G such that the angles at a vertex v in the sector $[e_{i+1}(v), e_{i-1}(v)]$ are labeled i (see Figure 12).

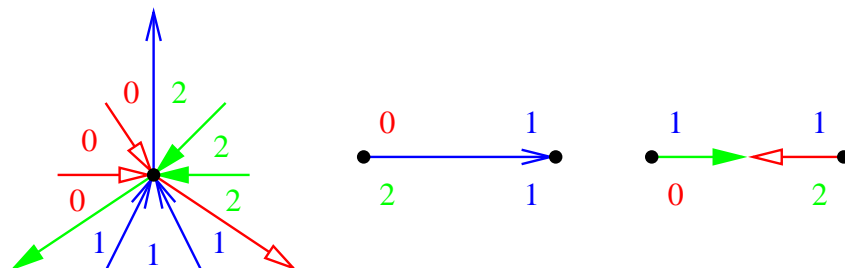


Figure 12: Angle labeling around vertices and edges

Lemma 16 *The angle labeling corresponding to a Schnyder wood of a toroidal map satisfies the following: the angles at each vertex and at each face form, in counterclockwise order, nonempty intervals of 0's, 1's and 2's.*

Proof. Clearly the property is true at each vertex by (T1). To prove that the property is true at each faces we count the number of color changes around vertices, faces and edges. This number of changes is denoted d . For a vertex v there are exactly three changes, so $d(v) = 3$ (see Figure 12). For an edge e , that can be either oriented in one or two direction, there are also exactly three changes, so $d(e) = 3$ (see Figure 12). Now consider a face F . Suppose we cycle counterclockwise around F , then an angle colored i is always followed by an angle colored i or $i + 1$. Consequently, $d(F)$ must be a multiple of three. Suppose that $d(F) = 0$, then all its angles are colored with one color i . In that case the cycle around face F would be completely oriented in counterclockwise order in color $i + 1$ (and in clockwise order in color $i - 1$). This cycle being contractible this would contradict Lemma 1. So $d(F) \geq 3$.

The sum of the changes around edges must be equal to the sum of the changes around faces and vertices. Thus $3m = \sum_e d(e) = \sum_v d(v) + \sum_F d(F) = 3n + \sum_F d(F)$. Euler's

formula gives $m = n + f$, so $\sum_F d(F) = 3f$ and this is possible only if $d(F) = 3$ for every face F . \square

There is no converse to Lemma 16. Figure 5 gives an example of a coloring and orientation of the edges of a toroidal triangulation not satisfying (T2) but where the angles at each vertex and at each face form, in counterclockwise order, nonempty intervals of 0's, 1's and 2's.

Let G be a toroidal map given with a Schnyder wood. By Lemma 15, G is essentially 3-connected, thus the dual G^* of G has no contractible loop and no homotopic multiple edges. Let \tilde{G} be a simultaneous drawing of G and G^* such that only dual edges intersect.

The *dual of the Schnyder wood* is the orientation and coloring of the edges of G^* obtained by the following method (see Figure 13 and 14): Let e be an edge of G and e^* the dual edge of e . If e is oriented in one direction only and colored i , then e^* is oriented in two direction, entering e from the right side in color $i - 1$ and from the left side in color $i + 1$ (the right side of e is the right side while following the orientation of e). Symmetrically, if e is oriented in two direction in color $i + 1$ and $i - 1$, then e^* is oriented in one direction only and colored i such that e is entering e^* from its right side in color $i - 1$.

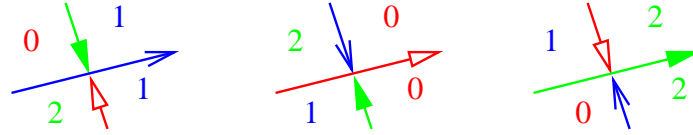


Figure 13: Rules for the dual Schnyder wood and angle labeling.

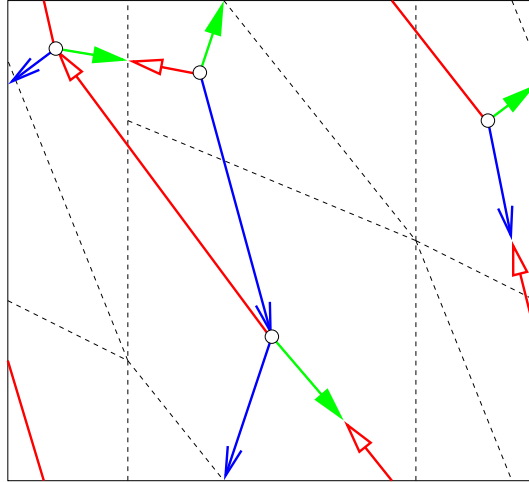


Figure 14: Dual Schnyder wood of the Schnyder wood of Figure 2.

Lemma 17 *Let G be a toroidal map. The dual of a Schnyder wood of a toroidal map G is a Schnyder wood of the dual G^* . Moreover we have :*

- (i) *On the simultaneous drawing \tilde{G} of G and G^* , the i -cycles of the dual Schnyder wood are homotopic to the i -cycles of the primal Schnyder wood and oriented in opposite direction.*
- (ii) *The dual of a Schnyder wood is of Type 2.i if and only if the primal Schnyder wood is of Type 2.i.*

Proof. In every face of \tilde{G} , there is exactly one angle of G and one angle of G^* . Thus a Schnyder angle labeling of G corresponds to an angle labeling of G^* . The dual of the Schnyder wood is defined such that an edge e is leaving F in color i if and only if the angle at F on the left of e is labeled $i - 1$ and the angle at F on the right of e is labeled $i + 1$, and such that an edge e is entering F in color i if and only if at least one of the angles at F incident to e is labeled i (see Figure 13). By Lemma 16, the angles at a face form, in counterclockwise order, nonempty intervals of 0's, 1's and 2's. Thus the edges around a vertex of G^* satisfy property (T1).

Consider \tilde{G} with the orientation and coloring of primal and dual edges.

Let C be a i -cycle of G^* . Suppose, by contradiction, that C is contractible. Let D be the disk delimited by C . Suppose by symmetry that C is going anticlockwise around D . Then all the edges of G that are dual to edges of C are entering D in color $i - 1$. Thus D contains a $i - 1$ -cycle of G , a contradiction to Lemma 1. Thus every monochromatic cycle of G^* is non contractible.

The dual of the Schnyder wood is defined in such a way that an edge of the primal and an edge of G^* of the same color never intersect in \tilde{G} . Thus the i -cycles of G^* are homotopic to i -cycles of G . Consider a i -cycle C_i (resp. C_i^*) of G (resp. G^*). The two cycles C_i and C_i^* are homotopic. By symmetry, we assume that the primal Schnyder wood is not of Type 2.(i-1). Let C_{i+1} be a $(i + 1)$ -cycle of G . The two cycles C_i and C_{i+1} are not homotopic and C_i is entering C_{i+1} on its left side. Thus, by Lemma 4, the two cycles C_i^* and C_{i+1} are not homotopic and by the dual rules C_i^* is entering C_{i+1} on its right side. So C_i and C_i^* are homotopic and going in opposite directions.

Suppose the Schnyder wood of G is of Type 1. Then two monochromatic cycles of G of different colors are not homotopic. Thus the same is true for monochromatic cycles of the dual. So (T2) is satisfied and the dual Schnyder wood is of Type 1.

Suppose now that the Schnyder wood of G is of Type 2. Assume by symmetry that it is of Type 2.i. Then all monochromatic cycles of color i and j , with $j \in \{i - 1, i + 1\}$, intersect. Now suppose, by contradiction, that there is a j -cycle C^* , with $j \in \{i - 1, i + 1\}$, that is not equal to a monochromatic cycle of color in $\{i - 1, i + 1\} \setminus \{j\}$. By symmetry we can assume that C^* is of color $i - 1$. Let C be the $(i - 1)$ -cycle of the primal that is the first on the right side of C^* in \tilde{G} . By definition of Type 2.i, C^{-1} is a $(i + 1)$ -cycle of G . Let R be the region delimited by C^* and C situated on the right side of C^* . Cycle C^* is not a $(i + 1)$ -cycle so there is at least one edge of color $i + 1$ leaving a vertex of

C^* . By (T1) in the dual, this edge is entering the interior of the region R . An edge of G^* of color $i + 1$ cannot intersect C and cannot enter C^* from its right side. So in the interior of the region R there is at least one $(i + 1)$ -cycle C_{i+1}^* of G^* . Cycle C_{i+1}^* is homotopic to C^* and going in opposite direction. If C_{i+1}^* is not a $(i + 1)$ -cycle, then we can define $R' \subsetneq R$ the region delimited by C_{i+1}^* and C situated on the left side of C_{i+1}^* and as before we can prove that there is a $(i - 1)$ -cycle of G^* in the interior of R' . So in any case, there is a $(i - 1)$ -cycle C_{i-1}^* of G^* in the interior of R . Let $R'' \subsetneq R$ be the region delimited by C^* and C_{i-1}^* situated on the right side of C^* . Clearly R'' does not contain C . Thus by definition of C , the region R'' does not contain any $(i - 1)$ -cycle of the primal. But R'' is non empty and contains at least one vertex v of G . The path $P_{i-1}(v)$ cannot leave R'' , a contradiction. So (T2) is satisfied and the dual Schnyder wood is of Type 2.i. \square

By Lemma 17, we have the following:

Theorem 7 *There is a bijection between Schnyder woods of a toroidal map and Schnyder woods of its dual.*

7 Existence of Schnyder woods for toroidal triangulations

In the plane, the proof of existence of Schnyder woods can be done without too much difficulty as the properties to be satisfied are only local. Property (P1) is clearly local and Property (P2) can be verified just by considering faces. In the toroidal case, things are much more complicated as property (T2) is truly global.

We prove existence of Schnyder wood by contracting edges until we obtain a graph with just one vertex. Then the graph can be decontracted step by step to obtain a Schnyder wood of the original graph. The two essentially 3-connected toroidal maps on one vertex are represented on Figure 15 with a toroidal Schnyder wood.

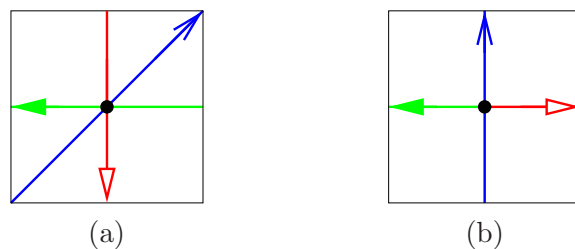


Figure 15: Schnyder woods of the two essentially 3-connected toroidal maps on one vertex.

Unfortunately, when there are edges oriented in two directions, we are not able to prove that property (T2) can be preserved during the decontraction process. Thus we prove the existence of Schnyder wood only for toroidal triangulations and leave the problem open for essentially 3-connected toroidal maps.

Given a toroidal triangulation G , the contraction of a non-loop-edge e of G is the operation consisting of continuously contracting e until merging its two ends, as shown on Figure 16. Note that only one edge of each pair of homotopic parallel edges is preserved (edges e_{wx} and e_{wy} on the figure).

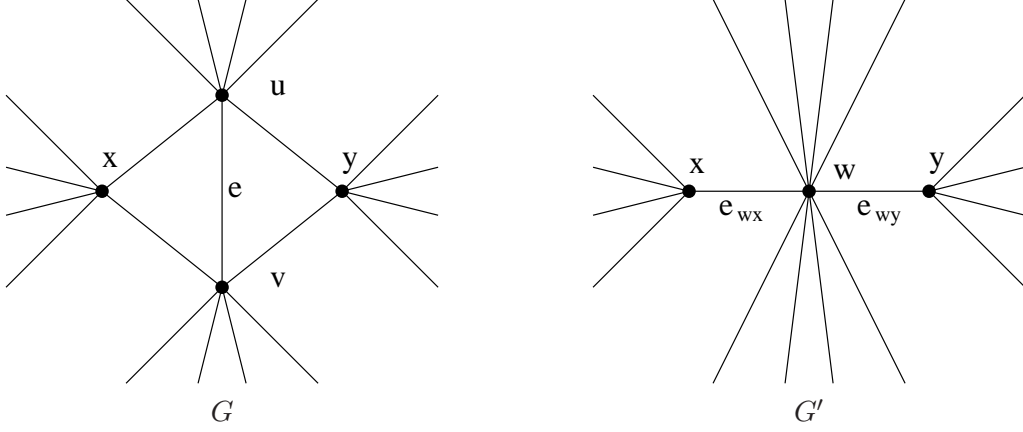


Figure 16: The contraction operation

Proof of Theorem 1. Suppose the theorem is false and let G be a toroidal triangulation, with no contractible loop and no homotopic multiple edges, that does not admit a Schnyder wood and that has the minimum number of vertices. The graph G is not a single vertex, otherwise it is easy to find a Schnyder wood of it (see Figure 15.a). Then one can find an edge e in G such that its contraction preserve the fact that G is a toroidal triangulation with no contractible loop and no homotopic multiple edges (see [20]). Let G' be the toroidal map obtained after contracting e .

By minimality, the graph G' has a Schnyder wood. Note that since G' is a toroidal triangulation, all edges are oriented in one direction only. We now prove how to extend the Schnyder wood of G' to a Schnyder wood of G . Let u, v be the two extremities of e and x, y the two vertices of G such that the two faces incident to e are $A = u, v, x$ and $B = v, u, y$ in clockwise order (see Figure 16). Note that u and v are distinct by definition of edge contraction but that x and y are not necessarily distinct, nor distinct from u and v . Let w be the vertex of G' resulting from the contraction of e and e_{wx}, e_{wy} be the two edges of G' represented on Figure 16. We say that a cycle C' of G' is *safe* if C' does not contain w .

There are different cases corresponding to the different possibilities of orientation and coloring of the edges e_{wx} and e_{wy} in G' . There should be 6 cases depending on if e_{wx} and e_{wy} are both entering w , both leaving w or one entering w and one leaving w (3 cases), multiplied by the coloring, both of the same or not (2 cases). The case where w has two edges leaving in the same color is impossible by (T1). So only 5 cases remain represented by figure $k.0$, for $k = 1, \dots, 5$, on Figures 17-19.

In each situation we show how one can orient and color the edges of G incident to the faces A and B to obtain a Schnyder wood of G from the Schnyder wood of G' (all the edges not incident to A or B are not modified). There can be several possible coloring of G represented by cases $k.\ell$, for $\ell \geq 1$, on Figures 17-19. In each case one can easily verify that property (T1) is still satisfied for every vertex. The difficult part is to show that property (T2) can be preserved by choosing the proper coloring and we will just focus on this part in the rest of the proof. By Lemma 9, we will just have to prove that (T2') is satisfied.

- *Case 1: e_{wx} and e_{wy} are entering w in different color.*

We can assume by symmetry that $e_{wx} = e_0(x)$ and $e_{wy} = e_2(y)$ (case 1.0 of Figure 17).

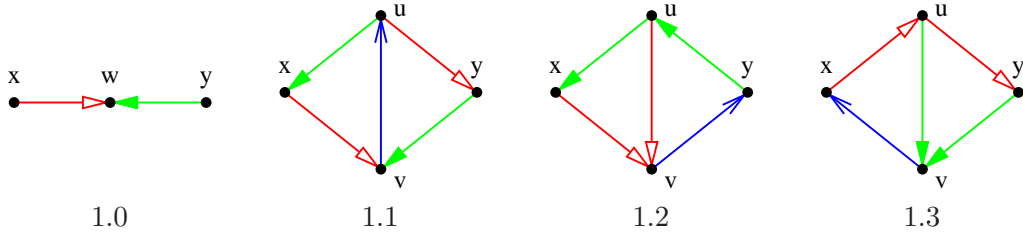


Figure 17: Decontraction rules for case 1

We now have a case analysis corresponding to whether there are monochromatic cycles of G' that are safe.

- ★ *Subcase 1. $\{0, 1, 2\}$: There are safe monochromatic cycles of colors $\{0, 1, 2\}$.*

Let C'_0, C'_1, C'_2 be safe monochromatic cycles of color 0, 1, 2 in G' . By Theorem 5, they pairwise intersect in G' . Apply the coloring 1.1 of Figure 17 on G . As C'_0, C'_1, C'_2 do not contain vertex w , they are not modified in G . Thus they still pairwise intersect in G . So (T2') is satisfied.

- ★ *Subcase 1. $\{0, 2\}$: There are safe monochromatic cycles of colors exactly $\{0, 2\}$.*

Let C'_0, C'_2 be safe monochromatic cycles of color 0, 2 in G' . Let C'_1 be a 1-cycle in G' . (Since C'_1 is not safe, it contains w , and so it is unique.) By Theorem 5, C'_0, C'_1, C'_2 pairwise intersect in G' . None of those intersections contain w as C'_0 and C'_2 do not contain w . By (T1), the cycle C'_1 enter w in the sector $]e_{wx}, e_{wy}[$ and leaves in the sector $]e_{wy}, e_{wx}[$. Apply the coloring 1.1 of Figure 17 on G . The cycle C'_1 is replaced by a new cycle $C_1 = C'_1 \setminus \{w\} \cup \{u, v\}$. The cycles C'_0, C'_1, C'_2 were intersecting outside w in G' so C'_0, C_1, C'_2 are intersecting in G . So (T2') is satisfied.

- ★ *Subcase 1. $\{1, 2\}$: There are safe monochromatic cycles of colors exactly $\{1, 2\}$.*

Let C'_1, C'_2 be safe monochromatic cycles of color 1, 2 in G' . Let C'_0 be a 0-cycle in G' . The cycles C'_0, C'_1, C'_2 pairwise intersect outside w . The cycle C'_0 enters w in the sector $[e_1(w), e_{wx}[$, $[e_{wx}, e_{wx}]$ or $]e_{wx}, e_2(w)[$. Apply the coloring 1.2 of Figure 17 on G . Depending on which of the three sectors C'_0 enters, it is replaced by one of the three following cycle $C_0 = C'_0 \setminus \{w\} \cup \{u, v\}$, $C_0 = C'_0 \setminus \{w\} \cup \{x, v\}$, $C_0 = C'_0 \setminus \{w\} \cup \{v\}$. In

any of the three possibilities, C_0, C'_1, C'_2 are intersecting in G . So (T2') is satisfied.

★ *Subcase 1. $\{0, 1\}$: There are safe monochromatic cycles of colors exactly $\{0, 1\}$.*

Completely symmetric to the case 1. $\{1, 2\}$.

★ *Subcase 1. $\{\}$: There are no safe monochromatic cycle.*

Let C'_0, C'_1, C'_2 be monochromatic cycle of color 0, 1, 2 in G' . They all pairwise intersect on w .

Suppose C'_0 is entering w in the sector $]e_{wx}, e_2(w)[$. Apply the coloring 1.3 of Figure 17 on G . The 0-cycle C'_0 is replaced by $C_0 = C'_0 \setminus \{w\} \cup \{v\}$ and thus contains v . Depending on which of the three sectors C'_2 enters, $[e_0(w), e_{wy}[$, $[e_{wy}, e_{wy}]$ or $]e_{wy}, e_1(w)[$, it is replaced by one of the three following cycle $C_2 = C'_2 \setminus \{w\} \cup \{v\}$, $C_2 = C'_2 \setminus \{w\} \cup \{y, v\}$, $C_2 = C'_2 \setminus \{w\} \cup \{u, v\}$. In any case, C_2 contains v . Finally, there must exists a 1-cycle C_1 in G (recall that every vertex has out degree 1 in color 1). Cycle C_1 has to contain u or v or both, otherwise it is a safe cycle of G' of color 1. Vertex u has no edge entering it in color 1 so C_1 does not contain u and thus it contains v . So C_0, C_1, C_2 all intersect on v and (T2') is satisfied.

The case where C'_2 is entering w in the sector $[e_0(w), e_{wy}[$ is completely symmetric and we apply the coloring 1.2 of Figure 17 on G .

It remains to deal with the case where C'_0 is entering w in the sector $[e_1(w), e_{wx}]$ and C'_2 is entering w in the sector $[e_{wy}, e_1(w)]$. Apply the coloring 1.1 of Figure 17 on G . Cycle C'_1 is replaced by $C_1 = C'_1 \setminus \{w\} \cup \{u, v\}$. Let C_0 be a 0-cycle of G of color 0. Cycle C_0 has to contain u or v or both, otherwise it is a safe cycle of G' of color 0. Suppose $C_0 \cap \{v, x\} = \{v\}$, then $C_0 \setminus \{v\} \cup \{w\}$ is a 0-cycle of G' entering w in the sector $]e_{wx}, e_2(w)[$, contradicting the assumption on C'_0 . Suppose C_0 contains u , then C_0 contains y , the extremity of the edge leaving u in color 0. So C_0 contains $\{v, x\}$ or $\{u, y\}$. Similarly C_2 contains $\{v, y\}$ or $\{u, x\}$. In any case C_0, C_1, C_2 pairwise intersect. So (T2') is satisfied.

★ *Subcase 1. $\{2\}$: There are safe monochromatic cycles of colors 2 only.*

Let C'_2 be a safe 2-cycle in G' . Let C'_0, C'_1 be monochromatic cycles of color 0, 1 in G' .

Suppose that there exists a path Q'_0 of color 0, from y to w , entering w in the sector $[e_1(w), e_{wx}]$, such that this path does not intersect C'_2 . Suppose also that there exists a path Q'_1 of color 1, from y to w , entering w in the sector $]e_{wx}, e_2(w)[$, such that this path does not intersect C'_2 . Let $C''_0 = Q'_0 \cup \{e_{wy}\}$ and $C''_1 = Q'_1 \cup \{e_{wy}\}$. By Lemma 1, C''_0, C''_1, C'_2 are not contractible. Both of C''_0, C''_1 does not intersect C'_2 so by Lemma 2, then are both homotopic to C'_2 . Thus by Lemma 3, cycles C''_0, C''_1 are homotopic to each other, contradicting Lemma 6 (with w, y, Q'_0, Q'_1). So we can assume that one of Q'_0 or Q'_1 as above does not exist.

Suppose that in G' , there does not exist a path of color 0, from y to w , entering w in the sector $[e_1(w), e_{wx}]$, such that this path does not intersect C'_2 . Apply the coloring 1.1 of Figure 17 on G . Cycle C'_1 is replaced by $C_1 = C'_1 \setminus \{w\} \cup \{u, v\}$, and intersect C'_2 . Let C_0 be a 0-cycle of G . Cycle C_0 has to contain u or v or both, otherwise it is a safe cycle

of G' of color 0. In any case it intersects C_1 . If C_0 contains v , then $C'_0 = C_0 \setminus \{v\} \cup \{w\}$ and so C_0 is intersecting C'_2 and (T2') is satisfied. Suppose now that C_0 does not contain v . Then C_0 contains u and y , the extremity of the edge leaving u in color 0. Let Q_0 be the part of C_0 consisting of the path from y to u . The path $Q'_0 = Q_0 \setminus \{u\} \cup \{w\}$ is from y to w and entering w in the sector $[e_1(w), e_{wx}]$. Thus by assumption Q'_0 intersects C'_2 . So C_0 intersects C'_2 and (T2') is satisfied.

Suppose now that in G' , there does not exist a path of color 1, from y to w , entering w in the sector $]e_{wx}, e_2(w)[$, such that this path does not intersect C'_2 . Apply the coloring 1.2 of Figure 17 on G . Depending on which of the three sectors C'_0 enters, $[e_1(w), e_{wx}[$, $[e_{wx}, e_{wx}]$ or $]e_{wx}, e_2(w)[$, it is replaced by one of the three following cycle $C_0 = C'_0 \setminus \{w\} \cup \{u, v\}$, $C_0 = C'_0 \setminus \{w\} \cup \{x, v\}$, $C_0 = C'_0 \setminus \{w\} \cup \{v\}$. In any of the three possibilities, C_0 contains v and intersect C'_2 . Let C_1 be a 1-cycle of G . Cycle C_1 has to contain u or v or both, otherwise it is a safe cycle of G' of color 1. Vertex u has no edge entering it in color 1 so C_1 does not contain u and thus it contains v and intersects C_0 . Then C_1 contains y , the extremity of the edge leaving v in color 1. Let Q_1 be the part of C_1 consisting of the path from y to v . The path $Q'_1 = Q_1 \setminus \{v\} \cup \{w\}$ is from y to w and entering w in the sector $]e_{wx}, e_2(w)[$. Thus by assumption Q'_1 intersects C'_2 . So C_1 intersects C'_2 and (T2') is satisfied.

★ *Subcase 1.{0}: There are safe monochromatic cycles of color 0 only.*

Completely symmetric to the case 1.{2}.

★ *Subcase 1.{1}: There are safe monochromatic cycles of color 1 only.*

Let C'_1 be a safe 1-cycle in G' . Let C'_0 and C'_2 be monochromatic cycles of color 0 and 2 in G' .

Suppose C'_0 is entering w in the sector $]e_{wx}, e_2(w)[$. Apply the coloring 1.3 of Figure 17 on G . The 0-cycle C'_0 is replaced by $C_0 = C'_0 \setminus \{w\} \cup \{v\}$ and thus contains v and still intersect C'_1 . Depending on which of the three sectors C'_2 enters, $[e_0(w), e_{wy}[$, $[e_{wy}, e_{wy}]$ or $]e_{wy}, e_1(w)[$, it is replaced by one of the three following cycle $C_2 = C'_2 \setminus \{w\} \cup \{v\}$, $C_2 = C'_2 \setminus \{w\} \cup \{y, v\}$, $C_2 = C'_2 \setminus \{w\} \cup \{u, v\}$. In any case, C_2 contains v and still intersect C'_1 . Cycle C_0 and C_2 intersect on v . So (T2') is satisfied.

The case where C'_2 is entering w in the sector $[e_0(w), e_{wy}[$ is completely symmetric and we apply the coloring 1.2 of Figure 17 on G .

It remains to deal with the case where C'_0 is entering w in the sector $[e_1(w), e_{wx}]$ and C'_2 is entering w in the sector $[e_{wy}, e_1(w)]$. Suppose that there exists a path Q'_0 of color 0, from y to w , entering w in the sector $[e_1(w), e_{wx}]$, such that this path does not intersect C'_1 . Suppose also that there exists a path Q'_2 of color 2, from x to w , entering w in the sector $[e_{wy}, e_1(w)]$, such that this path does not intersect C'_1 . Let $C''_0 = Q'_0 \cup \{e_{wy}\}$ and $C''_2 = Q'_2 \cup \{e_{wx}\}$. By Lemma 1, C''_0, C'_1, C''_2 are not contractible. Both of C''_0, C''_2 does not intersect C'_1 so by Lemma 2, then are both homotopic to C'_1 . Thus by Lemma 3, cycles C''_0, C''_2 are homotopic to each other. Thus by Lemma 7 (with w, x, y, Q'_0, Q'_2), we have $C''_0 = (C''_2)^{-1}$, contradicting the fact that there is no edges oriented in two direction in G' . So we can assume that one of Q'_0 or Q'_2 as above does not exist. By symmetry,

suppose that in G' , there does not exist a path of color 0, from y to w , entering w in the sector $[e_1(w), e_{wx}]$, such that this path does not intersect C'_1 . Apply the coloring 1.3 of Figure 17 on G . Depending on which of the two sectors C'_2 enters, $[e_{wy}, e_{wy}]$ or $]e_{wy}, e_1(w)[$, it is replaced by one of the two following cycle $C_2 = C'_2 \setminus \{w\} \cup \{y, v\}$, $C_2 = C'_2 \setminus \{w\} \cup \{u, v\}$. In any case, C_2 still intersect C'_1 . Let C_0 be a 0-cycle of G . Cycle C_0 has to contain u or v or both, otherwise it is a safe cycle of G' of color 0. Suppose C_0 does not contain u , then $C'_0 = C_0 \setminus \{v\} \cup \{w\}$ and C'_0 is not entering w in the sector $[e_1(w), e_{wx}]$, a contradiction. So C_0 contains u . Thus C_0 contains y , the extremity of the edge leaving u in color 0, and it intersects C_2 . Let Q_0 be the part of C_0 consisting of the path from y to u . The path $Q'_0 = Q_0 \setminus \{u\} \cup \{w\}$ is from y to w and entering w in the sector $[e_1(w), e_{wx}]$. Thus by assumption Q'_0 intersects C'_1 . So C_0 intersects C'_1 and (T2') is satisfied.

- *Case 2: e_{wx} and e_{wy} have the same color, one is entering w , the other is leaving w .*

We can assume by symmetry that $e_{wx} = e_1(x)$ and $e_{wy} = e_1(w)$ (case 2.0 of Figure 18).

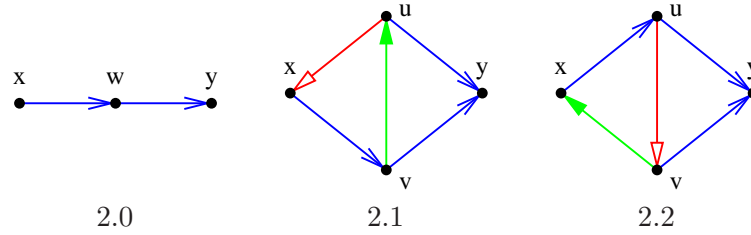


Figure 18: Decontraction possibilities for case 2

We again have a case analysis corresponding to whether there are monochromatic cycles of each color that are safe.

- ★ *Subcase 2. $\{0, 1, 2\}$: There are safe monochromatic cycles of colors $\{0, 1, 2\}$.*

Similar to case 1. $\{0, 1, 2\}$ by applying the coloring 2.1 of Figure 18 on G

- ★ *Subcase 2. $\{0, 2\}$: There are safe monochromatic cycles of colors exactly $\{0, 2\}$.*

Let C'_0, C'_2 be safe monochromatic cycles of color 0, 2 in G' . Let C'_1 be a 1-cycle in G' . Cycles C'_0, C'_2 still intersect in G . Apply the coloring 2.1 of Figure 18 on G . Depending on which of the three sectors C'_1 enters, $[e_2(w), e_{wx}]$, $[e_{wx}, e_{wx}]$ or $]e_{wx}, e_0(w)[$, it is replaced by one of the three following cycle $C_1 = C'_1 \setminus \{w\} \cup \{u, y\}$, $C_1 = C'_1 \setminus \{w\} \cup \{x, v, y\}$, $C_1 = C'_1 \setminus \{w\} \cup \{v, y\}$. In any of the three possibilities, C_1 still intersect both C'_0, C'_2 . So (T2') is satisfied.

- ★ *Subcase 2. $\{1, 2\}$: There are safe monochromatic cycles of colors exactly $\{1, 2\}$.*

Let C'_1, C'_2 be safe monochromatic cycles of color 1, 2 in G' . Let C'_0 be a 0-cycle in

G' . Cycles C'_1, C'_2 still intersect in G . Apply the coloring 2.2 of Figure 18 on G . Cycle C'_0 is replaced by $C_0 = C'_0 \setminus \{w\} \cup \{u, v\}$ and still intersect both C'_1, C'_2 . So (T2') is satisfied.

★ *Subcase 2.{0, 1}: There are safe monochromatic cycles of colors exactly {0, 1}.*

Completely symmetric to the case 2.{1, 2}.

★ *Subcase 2.{}: There are no safe monochromatic cycle.*

Let C'_0, C'_1, C'_2 be monochromatic cycle of color 0, 1, 2 in G' .

Suppose C'_1 is entering w in the sector $[e_2(w), e_{wx}]$. Apply the coloring 2.1 of Figure 18 on G . Depending on if C'_1 is entering w on e_{wx} or not, cycle C'_1 is replaced by $C_1 = C'_1 \setminus \{w\} \cup \{u, y\}$ or $C_1 = C'_1 \setminus \{w\} \cup \{x, v, y\}$. Cycle C'_0 is replaced by $C_0 = C'_0 \setminus \{w\} \cup \{u, x\}$. Cycle C'_2 is replaced by $C_2 = C'_2 \setminus \{w\} \cup \{u, v\}$. So C_0, C_1, C_2 all intersect on u . So (T2') is satisfied.

The case where C'_1 is entering w in the sector $]e_{wx}, e_1(w)]$ is completely symmetric and we apply the coloring 2.2 of Figure 18 on G .

★ *Subcase 2.{2}: There are safe monochromatic cycles of color 2 only.*

Let C'_2 be a safe 2-cycle in G' . Let C'_0, C'_1 be monochromatic cycles of color 0, 1 in G' .

Apply the coloring 2.2 of Figure 18 on G . Depending on which of the three sectors C'_1 enters, $[e_2(w), e_{wx}[$, $[e_{wx}, e_{wx}]$ or $]e_{wx}, e_0(w)]$, it is replaced by one of the three following cycle $C_1 = C'_1 \setminus \{w\} \cup \{u, y\}$, $C_1 = C'_1 \setminus \{w\} \cup \{x, u, y\}$, $C_1 = C'_1 \setminus \{w\} \cup \{v, y\}$. Cycle C'_0 is replaced by $C_0 = C'_0 \setminus \{w\} \cup \{u, v\}$. In any case, C_0 and C_1 intersects each other and intersect C'_2 . So (T2') is satisfied.

★ *Subcase 2.{0}: There are safe monochromatic cycles of color 0 only.*

Completely symmetric to the case 2.{0}.

★ *Subcase 2.{1}: There are safe monochromatic cycles of color 1 only.*

Let C'_1 be a safe 1-cycle in G' . Let C'_0, C'_2 be monochromatic cycles of color 0, 2 in G' . Suppose that there exists a path Q'_0 of color 0, from x to w , that does not intersect C'_1 . Suppose also that there exists a path Q'_2 of color 2, from x to w , that does not intersect C'_1 . Let $C''_0 = Q'_0 \cup \{e_{wx}\}$ and $C''_2 = Q'_2 \cup \{e_{wx}\}$. By Lemma 1, C''_0, C'_1, C''_2 are not contractible. Both of C''_0, C''_2 does not intersect C'_1 so by Lemma 2, then are both homotopic to C'_1 . Thus by Lemma 3, cycles C''_0, C''_2 are homotopic to each other, contradicting Lemma 6 (with w, x, Q'_0, Q'_2). So we can assume that one of Q'_0 or Q'_2 as above does not exist. By symmetry, suppose that in G' , there does not exist a path of color 0, from x to w , that does not intersect C'_1 . Apply the coloring 2.1 of Figure 18 on G . Cycle C'_2 is replaced by $C_2 = C'_2 \setminus \{w\} \cup \{u, v\}$ and so intersects C'_1 . Let C_0 be a 0-cycle of G . Cycle C_0 has to contain u or v or both, otherwise it is a safe cycle of G' of color 0. Vertex v has no edge entering it in color 0, so C_0 does not contain v and so it contains u and x , the extremity of the edge leaving u in color 0. Thus C_0 intersect C_2 . Let Q_0 be the part of C_0 consisting of the path from x to u . The path $Q'_0 = Q_0 \setminus \{u\} \cup \{w\}$ is from x to w , thus by assumption Q'_0 intersects C'_1 . So C_0 intersects C'_1 and (T2') is satisfied.

• *Remaining cases:*

The proof is simpler for the remaining cases (cases 3.0, 4.0, 5.0 on Figure 19). For each situation, there is only one way to extend the coloring to G in order to preserve (T1) and this coloring also preserve (T2'). In each coloring of G , case 3.1, 4.1, 5.1 on Figure 19, every non safe monochromatic cycle C' is replaced by a cycle C with $C \supseteq C' \setminus \{w\} \cup \{u\}$. Thus all non safe cycles intersects on u and a non safe cycle of color i intersect all safe cycles of colors $i - 1$ and $i + 1$. So (T2') is always satisfied.

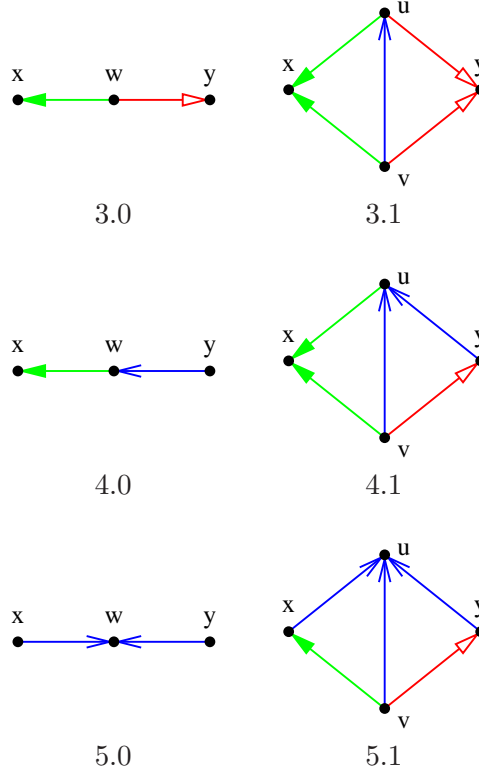


Figure 19: Decontraction rules for cases 3, 4, 5

□

Here is a remark about how to algorithmically compute a Schnyder wood for a toroidal triangulation. Instead of looking carefully at the proof of Theorem 1 to know which coloring of the decontracted graph has to be chosen. One can try all of them and check after if the obtained colorings are Schnyder woods. To do so, one just as to check if (T2') is satisfied. Checking that (T2') is satisfied can be done by the following method: start from any vertex v , walk along $P_0(v), P_1(v), P_2(v)$ and mark the three monochromatic cycles C_0, C_1, C_2 reached by the three paths P_i . By Lemma 9 and

Theorem 5, (T2') is satisfied if and only if the cycles C_0, C_1, C_2 pairwise intersect. So one just has to check if C_0, C_1, C_2 pairwise intersect.

For essentially 3-connected toroidal maps the method of the proof of Theorem 1 does not work. It is easy to contract the graph and then decontract it to find a coloring of G that satisfies (T1) (although if the number of different cases to consider is large) but it is harder to show how to preserve (T2). On the example of Figure 20, it is not possible to decontract the graph G' (Figure 20.(a)), and extend its Schnyder wood to G (Figure 20.(b)) without modifying the edges that are not incident to the contracted edge e . Indeed, if we keep the edges non-incident to e unchanged, there are only two possible ways to extend the coloring in order to preserve (T1) (represented on Figures 20.(c) and 20.(d)), but none of them fulfills (T2). On Figure 20.(c), there is a cycle of color 2, not intersecting a cycle of color 1. On Figure 20.(d), there is a cycle of color 1, not intersecting a cycle of color 2.

Even if the same proof of Theorem 1 does not work on essentially 3-connected toroidal maps, we propose the following conjecture:

Conjecture 1 *A toroidal map admits a Schnyder wood if and only if it is essentially 3-connected.*

Open problems related to Conjecture 1 consist in characterizing which essentially 3-connected toroidal maps have only Schnyder woods of Type 1 or only Schnyder woods of Type 2. We believe that essentially 3-connected toroidal maps having only Schnyder woods of Type 2 form a very basic class of toroidal maps depicted on Figure 21

The existence of Schnyder wood for toroidal triangulations implies the following :

Theorem 8 *A toroidal triangulation contains three non contractible and non homotopic cycles that are pairwise edge-disjoint.*

Proof. One just has to apply Theorem 1 to obtain a Schnyder wood of Type 1 and then, for each color i , choose arbitrarily a i -cycle. These cycles are edge-disjoint as there is no bi-oriented edges in Schnyder woods of toroidal triangulations. \square

Fijavz [13] proved a stronger result than Theorems 8 when restricted to toroidal triangulations with no loop and no multiple edges. Recall that in this paper we are less restrictive as we allow non contractible loops and non homotopic multiple edges.

Theorem 9 ([13]) *A simple toroidal triangulation contains three non contractible and non homotopic cycles that all intersect on one vertex and that are pairwise disjoint otherwise.*

Theorem 9 is not true for (general) toroidal triangulation (see example on Figure 22).

Theorem 9 can be used to obtain Schnyder woods where any pair of monochromatic cycles of different colors intersect only once.

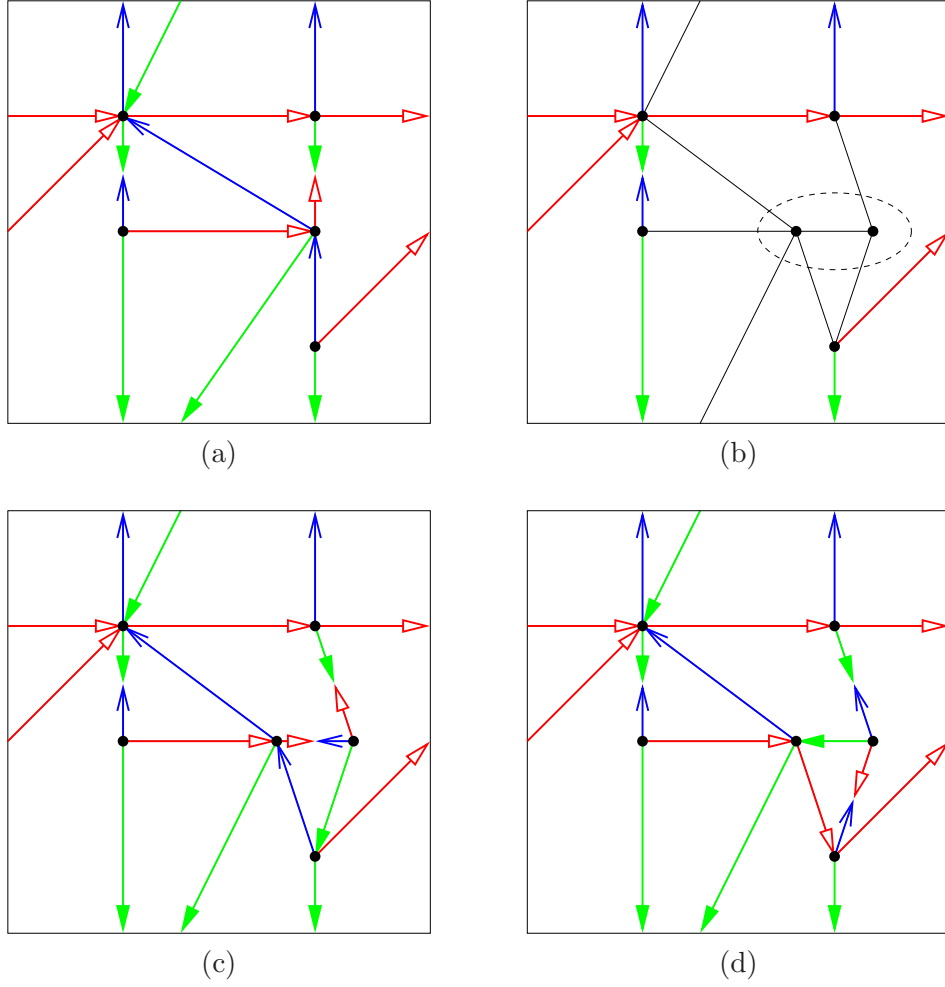


Figure 20: (a) A Schnyder wood of the graph G' obtained by the contraction of an edge of the graph G , (b) the graph G , (c) and (d) two possible ways to extend the coloring in order to preserve (T1), but none of them fulfills (T2).

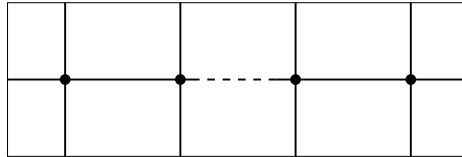


Figure 21: A family of toroidal maps having only Schnyder woods of Type 2

Theorem 10 *A simple toroidal triangulation admits a Schnyder wood with three monochromatic cycles of different colors all intersecting on one vertex and that are pairwise disjoint otherwise.*

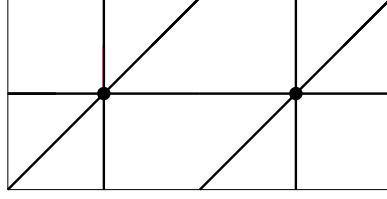


Figure 22: A toroidal triangulation that does not contain three non contractible and non homotopic cycles that all intersect on one vertex and that are pairwise disjoint otherwise.

Proof. Let G be a simple toroidal triangulation. By Theorem 9, let C_0, C_1, C_2 be three non contractible and non homotopic cycles of G that all intersect on one vertex x and that are pairwise disjoint otherwise. By eventually shortening the cycles C_i , we can assume that the three cycles C_i are homotopically chordless (i.e. there is no edge between two vertices of C_i that can be continuously transformed into a part of C_i). By symmetry, we can assume that the six edges e_i, e'_i of the cycles C_i incident to x appear around x in the counterclockwise order $e_0, e'_2, e_1, e'_0, e_2, e'_1$. The cycles C_i divide G into two regions, denoted R_1, R_2 such that R_1 is the region situated in the sector $[e_0, e'_2]$ of x and R_2 is the region situated in the sector of $[e'_2, e_1]$ of x . Let G_i be the subgraph of G contained in the region R_i (including the three cycles C_i). Let G'_1 (resp. G'_2) be the graph obtained from G_1 (resp. G_2) by replacing x by three vertices x_1, x_2, x_3 , such that x_i is incident to the edges in the sector $[e_{i+1}, e'_i]$ (resp. $[e'_i, e_{i-1}]$). The cycles C_i being homotopically chordless, the two graphs G'_1 and G'_2 are internally 3-connected planar maps (for vertices x_i chosen on their outer face). The vertices x_0, x_1, x_2 appear in counterclockwise order on the outer face of G'_1 and G'_2 . By a result of Miller [19] (see also [7, 8]), the two graphs G'_i admit planar Schnyder woods rooted at x_0, x_1, x_2 . Orient and color the edges of G that intersect the interior of R_i by giving them the same orientation and coloring as in a planar Schnyder wood of G'_i . Orient and color the cycle C_i in color i such that it is entering x by edge e'_i and leaving x by edge e_i . We claim that the orientation and coloring that is obtained is a toroidal Schnyder wood of G .

Clearly, any interior vertex of the region R_i satisfies (T1). Let us show that (T1) is also satisfied for any vertex v of a cycle C_i distinct from x . In a Schnyder wood of G'_1 , the cycle C_i is oriented in two direction, from x_{i-1} to x_i in color i and from x_i to x_{i-1} in color $i - 1$. Thus the edge leaving v in color $i + 1$ is an inner edge of G'_1 and vertex v has no edges entering in color $i + 1$. Symmetrically, in G'_2 the edge leaving v in color $i - 1$ is an inner edge of G'_2 and vertex v has no edges entering in color $i - 1$. Then one can paste G'_1 and G'_2 along C_i , orient C_i in color i and see that v satisfies (T1). The definition the G'_i , and the orientation of the cycles is done so that x satisfies (T1). The cycles C_i being pairwise intersecting, (T2') is satisfied, so by Lemma 9, (T2) is satisfied. \square

Note that in the Schnyder wood obtained by Theorem 10, we do not know if there are several monochromatic cycles of one color or not. So given any three monochromatic

cycles of different color, they might not all intersect on one vertex. But for any two monochromatic cycles of different color, we know that they intersect just once. We wonder whether Theorem 10 can be strengthened in such a way that there is just one monochromatic cycle per color.

A nonempty family \mathcal{R} of linear orders on the vertex set V of a simple graph G is called a *realizer* of G if for every edge e , and every vertex x not in e , there is some order $<_i \in \mathcal{R}$ so that $y <_i x$ for every $y \in e$. The *Dushnik-Miller dimension* [6] of G , is defined as the least positive integer t for which G has a realizer of cardinality t . Realizers are usually used on finite graphs, but here we allow G to be an infinite simple graph.

Schnyder wood where originally defined by Schnyder [24] to proved that a finite planar graph G has Dushnik-Miller dimension at most 3. A consequence of Theorem 1 is an analogous result for the universal cover of a toroidal graph:

Theorem 11 *The universal cover of a toroidal graph has Dushnik-Miller dimension at most three.*

Proof. By eventually adding edges to G we may assume that G is a toroidal triangulation. By Theorem 1, the graph G admits a Schnyder wood. For $i \in \{0, 1, 2\}$, let $<_i$ be the order induced by the inclusion of the regions R_i in G^∞ . That is $u <_i v$ if and only if $R_i(u) \subsetneq R_i(v)$. Let $<'_i$ be any linear extension of $<_i$ and consider $\mathcal{R} = \{<'_0, <'_1, <'_2\}$. Let e be any edge of G^∞ and v be any vertex of G^∞ not in e . Edge e is in a region $R_i(v)$ for some i , thus $R_i(u) \subseteq R_i(v)$ for every $u \in e$. As there is no edges oriented in two directions in a Schnyder wood of a toroidal triangulation, we have $R_i(u) \neq R_i(v)$ and so $u <_i v$. Thus \mathcal{R} is a realizer of G^∞ . \square

8 Orthogonal surfaces

Given two points $u = (u_0, u_1, u_2)$ and $v = (v_0, v_1, v_2)$ in \mathbb{R}^3 , we note $u \vee v = (\max(u_i, v_i))_{i=0,1,2}$ and $u \wedge v = (\min(u_i, v_i))_{i=0,1,2}$. We define an order \geq among the points in \mathbb{R}^3 , in such a way that $u \geq v$ if $u_i \geq v_i$ for $i = 0, 1, 2$.

Given a set \mathcal{V} of pairwise incomparable elements in \mathbb{R}^3 , we define the set of vertices that dominates \mathcal{V} as $\mathcal{D}_{\mathcal{V}} = \{u \in \mathbb{R}^3 \mid \exists v \in \mathcal{V} \text{ such that } u \geq v\}$. The *orthogonal surface* $\mathcal{S}_{\mathcal{V}}$ generated by \mathcal{V} is the boundary of $\mathcal{D}_{\mathcal{V}}$. (Note that orthogonal surfaces are well defined even when \mathcal{V} is an infinite set.) If $u, v \in \mathcal{V}$ and $u \vee v \in \mathcal{S}_{\mathcal{V}}$, then $\mathcal{S}_{\mathcal{V}}$ contains the union of the two line segment joining u and v to $u \vee v$. Such arcs are called *elbow geodesic*. The *orthogonal arc* of $v \in \mathcal{V}$ in the direction of the standard basis vector e_i is the intersection of the ray $v + \lambda e_i$ with $\mathcal{S}_{\mathcal{V}}$.

Let G be a planar map. A *geodesic embedding* of G on the orthogonal surface $\mathcal{S}_{\mathcal{V}}$ is a drawing of G on $\mathcal{S}_{\mathcal{V}}$ satisfying the following:

(D1) There is a bijection between the vertices of G and \mathcal{V} .

- (D2) Every edge of G is an elbow geodesic.
- (D3) Every orthogonal arc in \mathcal{S}_V is part of an edge of G .
- (D4) There are no crossing edges in the embedding of G on \mathcal{S}_V .

Miller [19] (see also [8, 11]) proved that a geodesic embedding of a planar map G on an orthogonal surface \mathcal{S}_V induces a Schnyder wood of G . The edges of G are colored with the direction of the orthogonal arc contained in the edge. An orthogonal arc intersecting the ray $v + \lambda e_i$ corresponds to the edge leaving v in color i . Edges represented by two orthogonal arcs corresponds to edges oriented in two directions.

Conversely, it has been proved that a Schnyder wood of a planar map G can be used to obtain a geodesic embedding of M . Let G be a planar map given with a Schnyder wood. The method is the following (see [8] for more details): For every vertex v , one can divide G into the three regions bounded by the three monochromatic path going out from v . The *region vector* associated to v is the vector obtained by counting the number of faces in each of these three regions. The mapping of each vertex on its region vector gives the geodesic embedding. (Note that in this approach, the vertices are all mapped on the same plane as the sum of the coordinates of each region vector is equal to the total number of inner faces of the map.)

Our goal is to generalize geodesic embedding to the torus. More precisely, we want to represent the universal cover of a toroidal map on an infinite and periodic orthogonal surface.

Let G be a toroidal map. Consider any flat torus representation of G in a parallelogram P . The graph G^∞ is obtained by replicating P to tile the plane. Given any of these parallelograms Q , let Q^{top} (resp. Q^{right}) be the copy of P just above (resp. on the right of) Q . Given a vertex v in Q , we note v^{top} (resp. v^{right}) its copies in Q^{top} (resp. Q^{right}).

A mapping of the vertices of G^∞ in \mathbb{R}^d is *periodic* with respect to vectors S and S' of \mathbb{R}^d , if there exists a flat torus representation P of G such that for any vertex v of G^∞ , vertex v^{top} is mapped on $v + S$ and v^{right} is mapped on $v + S'$. A *geodesic embedding* of a toroidal map G is a geodesic embedding of G^∞ on \mathcal{S}_{V^∞} , where V^∞ is a periodic mapping of G^∞ with respect to two non colinear vectors (see example of Figure 23).

Like in the plane Schnyder woods can be used to obtain geodesic embeddings of toroidal maps. For that purpose, we need to generalize the region vector method used in the plane. The idea is to use the regions $R_i(v)$ to compute the coordinates of the vertex v of G^∞ . The problem is that contrarily to the planar case, these regions are unbounded and contains an infinite number of faces. The method is thus generalized by the following.

Let G be a toroidal map, given with a Schnyder wood and a flat torus representation in a parallelogram P .

The *size* of the region $R(C_i^j, C_i^{j+1})$ of G , denoted $f_i^j = |R(C_i^j, C_i^{j+1})|$, is equal to the number of faces in $R(C_i^j, C_i^{j+1})$. Remark that for each color, we have $\sum_{j=0}^{k_i-1} f_i^j$ equals the

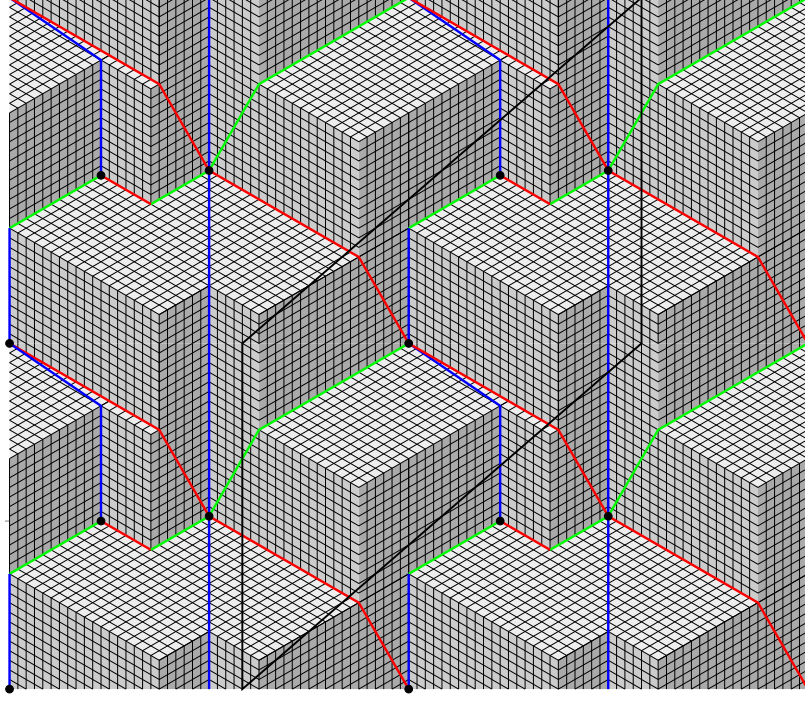


Figure 23: Geodesic embedding of the toroidal map of Figure 2.

total number of faces f of G . If L and L' are consecutive i -lines of G^∞ with $L \in \mathcal{L}_i^j$ and $L' \in \mathcal{L}_i^{j+1}$, then the *size* of the (unbounded) region $R(L, L')$, denoted $|R(L, L')|$, is equal to f_i^j . If L and L' are any i -lines, the *size* of the (unbounded) region $R(L, L')$, denoted $|R(L, L')|$, is equal to the sum of the size of all the regions delimited by consecutive i -lines inside $R(L, L')$.

For each color i , choose arbitrarily a i -line L_i^* in \mathcal{L}_i^0 that is used as an origin for i -lines. The *positive side* of a i -line is define (like in G) as the right side while “walking” along the directed path by following the orientation of the edges colored i . Given a i -line L , we define the value $f_i(L)$ of L as follows: $f_i(L) = |R(L, L_i^*)|$ if L is on the positive side of L_i^* and $f_i(L) = -|R(L, L_i^*)|$ otherwise.

Consider two vertices u, v such that $L_{i-1}(u) = L_{i-1}(v)$ and $L_{i+1}(u) = L_{i+1}(v)$. Even if the two regions $R_i(u)$ and $R_i(v)$ are unbounded, their *difference* is bounded. Let $d_i(u, v)$ be the number of faces in $R_i(u) \setminus R_i(v)$ minus the number of faces in $R_i(v) \setminus R_i(u)$. By Lemma 14, let $z_i(v)$ be a vertex on the intersection of the two lines $L_{i-1}(v)$ and $L_{i+1}(v)$. Let N be a constant $\geq n$ (in this section we can have $N = n$ but in Section 10 we need to choose N much bigger). We now define the region vector of the vertices of G^∞ , that is a mapping of these vertices in \mathbb{R}^3 .

Definition 4 *The i -th coordinate of the region vector of a vertex v of G^∞ is equal to $v_i = d_i(v, z_i(v)) + N \times (f_{i+1}(L_{i+1}(v)) - f_{i-1}(L_{i-1}(v)))$ (see Figure 24).*

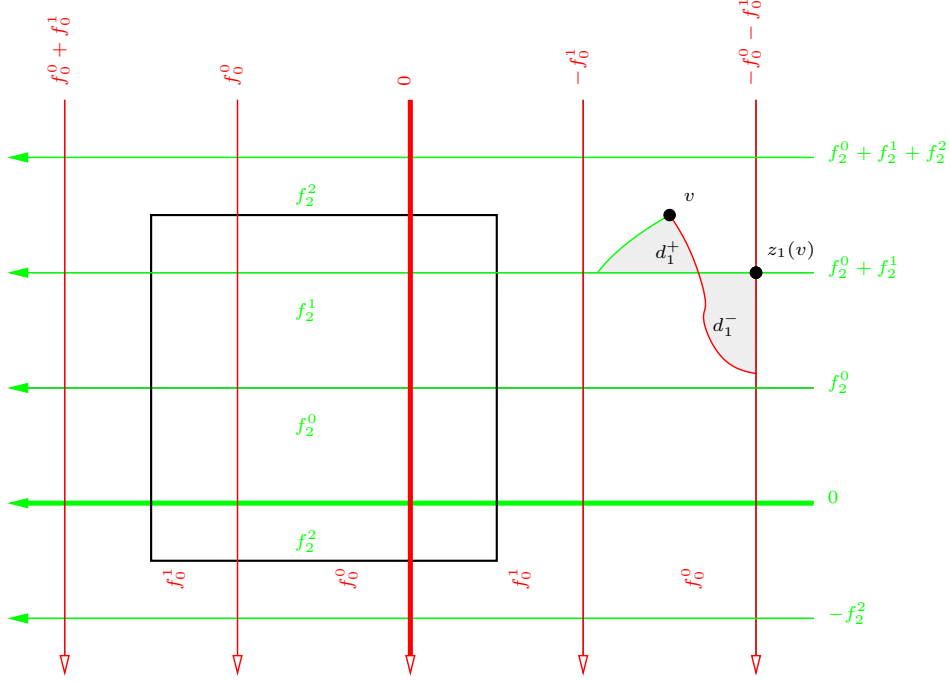


Figure 24: Coordinate 1 of vertex v , v_1 , is equal to the number of faces in the region d_1^+ , minus the number of faces in the region d_1^- , plus N times $(f_2^0 + f_2^1) - (-f_0^0 - f_0^1)$.

Lemma 18 *The sum of the coordinates of a vertex v equals the number of faces in the bounded region delimited by the lines $L_0(v)$, $L_1(v)$ and $L_2(v)$ if the Schnyder wood is of Type 1 and this sum equals zero if the Schnyder wood is of Type 2.*

Proof. We have $v_0 + v_1 + v_2 = d_0(v, z_0(v)) + d_1(v, z_1(v)) + d_2(v, z_2(v)) = \sum_i (|R_i(v) \setminus R_i(z_i(v))| - |R_i(z_i(v)) \setminus R_i(v)|)$. We use the characteristic function $\mathbf{1}$ to deal with infinite regions. We note $\mathbf{1}(R)$, the function defined on the faces of G^∞ that has value 1 on each face of region R and 0 elsewhere. Given a function $g : R \rightarrow \mathbb{Z}$, we note $|g| = \sum_{F \in R} g(F)$ (when the sum is finite). Thus $\sum_i v_i = \sum_i (|\mathbf{1}(R_i(v) \setminus R_i(z_i(v)))| - |\mathbf{1}(R_i(z_i(v)) \setminus R_i(v))|) = |\sum_i (\mathbf{1}(R_i(v) \setminus R_i(z_i(v))) - \mathbf{1}(R_i(z_i(v)) \setminus R_i(v)))|$. Now we compute $g = \sum_i (\mathbf{1}(R_i(v) \setminus R_i(z_i(v))) - \mathbf{1}(R_i(z_i(v)) \setminus R_i(v)))$. We have:

$$g = \sum_i (\mathbf{1}(R_i(v) \setminus R_i(z_i(v))) + \mathbf{1}(R_i(v) \cap R_i(z_i(v))) - \mathbf{1}(R_i(z_i(v)) \setminus R_i(v)) - \mathbf{1}(R_i(v) \cap R_i(z_i(v))))$$

As $R_i(v) \setminus R_i(z_i(v))$ and $R_i(z_i(v)) \setminus R_i(v)$ are disjoint from $R_i(v) \cap R_i(z_i(v))$, we have

$$g = \sum_i (\mathbf{1}(R_i(v)) - \mathbf{1}(R_i(z_i(v)))) = \sum_i \mathbf{1}(R_i(v)) - \sum_i \mathbf{1}(R_i(z_i(v)))$$

The interior of the region $R_i(v)$ being disjoint and spanning the whole plane \mathbb{P} (by definition), we have $\sum_i \mathbf{1}(R_i(v)) = \mathbf{1}(\cup_i (R_i(v))) = \mathbf{1}(\mathbb{P})$. Moreover the region $R_i(z_i(v))$

are also disjoint and $\sum_i \mathbf{1}(R_i(z_i(v))) = \mathbf{1}(\cup_i (R_i(z_i(v)))) = \mathbf{1}(\mathbb{P} \setminus T)$ where T is the bounded region delimited by the lines $L_0(v)$, $L_1(v)$ and $L_2(v)$. So $g = \mathbf{1}(\mathbb{P}) - \mathbf{1}(\mathbb{P} \setminus T) = \mathbf{1}(T)$. And thus $\sum_i v_i = |g| = |\mathbf{1}(T)|$. \square

Lemma 18 shows that if the Schnyder wood is of Type 1, then the set of points are not necessarily coplanar like in the planar case (see [12]), but all the copies of a vertex lies on the same plane (the bounded region delimited by the lines $L_0(v)$, $L_1(v)$ and $L_2(v)$) has the same number of faces for any copies of a vertex v). Surprisingly, for Schnyder woods of Type 2, all the points are coplanar.

For each color i , let c_i (resp. c'_i), be the algebraic number of times a i -cycle is traversing the vertical (resp. horizontal) side of the parallelogram P (that was the parallelogram containing the flat torus representation of G) from right to left (resp. from top to bottom). This number increases by one each time a monochromatic cycle traverses the side in the given direction and decreases by one when it traverses in the other way. Let S and S' be the two vectors of \mathbb{R}^3 with coordinates $S_i = N(c_{i+1} - c_{i-1})f$ and $S'_i = N(c'_{i+1} - c'_{i-1})f$. Note that $S_0 + S_1 + S_2 = 0$ and $S'_0 + S'_1 + S'_2 = 0$

Lemma 19 *The mapping is periodic with respect to S and S' .*

Proof. Let v be any vertex of G^∞ . Then $v_i^{top} - v_i = N(f_{i+1}(L_{i+1}(v^{top})) - f_{i+1}(L_{i+1}(v))) - N(f_{i-1}(L_{i-1}(v^{top})) - f_{i-1}(L_{i-1}(v))) = N(c_{i+1} - c_{i-1})f$. So $v^{top} = v + S$. Similarly $v^{right} = v + S'$. \square

For each color i , let γ_i be the integer such that two monochromatic cycles of G of respective colors $i - 1$ and $i + 1$ intersect exactly γ_i times, with the convention that $\gamma_i = 0$ if the Schnyder wood is of Type 2.i. By Lemma 5, γ_i is properly defined and do not depend on the choice of the monochromatic cycles. Note that if the Schnyder wood is of Type 2.i, then $\gamma_{i-1} = \gamma_{i+1}$ and if the Schnyder wood is not of Type 2.i, then $\gamma_i \neq 0$. Let $\gamma = \max(\gamma_0, \gamma_1, \gamma_2)$. Let $Z_0 = ((\gamma_1 + \gamma_2)Nf, -\gamma_1Nf, -\gamma_2Nf)$ and $Z_1 = (-\gamma_0Nf, (\gamma_0 + \gamma_2)Nf, -\gamma_2Nf)$ and $Z_2 = (-\gamma_0Nf, -\gamma_1Nf, (\gamma_0 + \gamma_1)Nf)$.

Lemma 20 *For any vertex u , we have $\{u + k_0Z_0 + k_1Z_1 + k_2Z_2 \mid k_0, k_1, k_2 \in \mathbb{Z}\} \subseteq \{u + kS + k'S' \mid k, k' \in \mathbb{Z}\}$.*

Proof. Let u, v be two copies of the same vertex, such that v is the first copy of u in the direction of $L_0(u)$. (That is $L_0(u) = L_0(v)$ and on the path $P_0(u) \setminus P_0(v)$ there is no two copies of the same vertex.) Then $v_i - u_i = N(f_{i+1}(L_{i+1}(v)) - f_{i+1}(L_{i+1}(u))) - N(f_{i-1}(L_{i-1}(v)) - f_{i-1}(L_{i-1}(u)))$. We have $|R(L_0(v), L_0(u))| = 0$, $|R(L_1(v), L_1(u))| = \gamma_2f$ and $|R(L_2(v), L_2(u))| = \gamma_1f$. So $v_0 - u_0 = N(\gamma_1 + \gamma_2)f$ and $v_1 - u_1 = -N\gamma_1f$ and $v_2 - u_2 = -N\gamma_2f$. So $v = u + Z_0$. Similarly for the other colors. So the first copy of u in the direction of $L_i(u)$ is equal to $u + Z_i$. \square

Lemma 21 *We have $\dim(Z_0, Z_1, Z_2) = 2$ and if the Schnyder wood is not of Type 2.i, then $\dim(Z_{i-1}, Z_{i+1}) = 2$.*

Proof. We have $\gamma_0 Z_0 + \gamma_1 Z_1 + \gamma_2 Z_2 = 0$ and so $\dim(Z_0, Z_1, Z_2) \leq 2$. We can assume by symmetry that the Schnyder wood is not of Type 2.1 and so $\gamma_1 \neq 0$. Thus $Z_0 \neq 0$ and $Z_2 \neq 0$. Suppose by contradiction that $\dim(Z_0, Z_2) = 1$. Then there exist $\alpha \neq 0$, $\beta \neq 0$, such that $\alpha Z_0 + \beta Z_2 = 0$. The sum of this equation for the coordinates 0 and 2 gives $(\alpha + \beta)\gamma_1 = 0$ and thus $\alpha = -\beta$. Then the equation for coordinate 0 gives $\gamma_0 + \gamma_1 + \gamma_2 = 0$ contradicting the fact that $\gamma_1 > 1$ and $\gamma_0, \gamma_2 \geq 0$. \square

Lemma 22 *The vectors S, S' are not colinear.*

Proof. By Lemma 20, the set $\{u + k_0 Z_0 + k_1 Z_1 + k_2 Z_2 \mid k_0, k_1, k_2 \in \mathbb{Z}\}$ is a subset of $\{u + kS + k'S' \mid k, k' \in \mathbb{Z}\}$. By Lemma 21, we have $\dim(Z_0, Z_1, Z_2) = 2$, thus $\dim(S, S') = 2$. \square

Lemma 23 *If u, v are two distinct vertices such that v is in $L_{i-1}(v)$, u is in $P_{i-1}(v)$, both u and v are in the region $R(L_{i+1}(u), L_{i+1}(v))$ and $L_{i+1}(u)$ and $L_{i+1}(v)$ are two consecutive $(i+1)$ -lines with $L_{i+1}(u) \in \mathcal{L}_{i+1}^j$ (see Figure 25). Then $d_i(z_i(v), v) + d_i(u, z_i(u)) < (n-1) \times f_{i+1}^j$.*

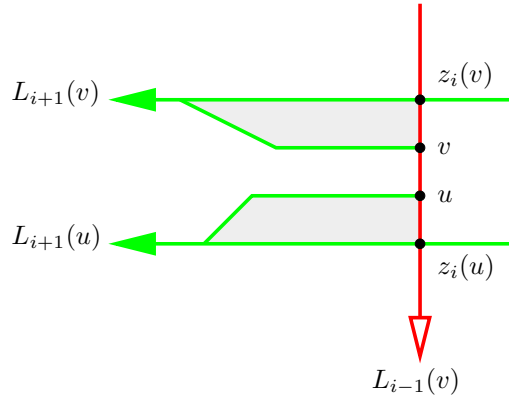


Figure 25: The gray area, corresponding to the quantity $d_i(z_i(v), v) + d_i(u, z_i(u))$, has size bounded by $(n-1) \times f_{i+1}^j$.

Proof. Let $Q_{i+1}(v)$ the subpath of $P_{i+1}(v)$ between v and $L_{i+1}(v)$ (maybe $Q_{i+1}(v)$ has length 0 if $v = z_i(v)$). Let $Q_{i+1}(u)$ the subpath of $P_{i+1}(u)$ between u and $L_{i+1}(u)$ (maybe $Q_{i+1}(u)$ has length 0 if $u = z_i(u)$). The path $Q_{i+1}(v)$ cannot contain two different copies of a vertex of G , otherwise $Q_{i+1}(v)$ will correspond to a non contractible cycle of G and thus will contain an edge of $L_{i+1}(v)$. So the length of $Q_{i+1}(v)$ is $\leq n-1$.

The total number of times a copy of a given face of G can appear in the region $R_i(z_i(v)) \setminus R_i(v)$ (corresponding to $d_i(z_i(v), v)$) is less or equal to the number of time that $Q_{i+1}(v)$ intersects a copy of $L_{i-1}(v)$. Each of these intersections corresponds to a different vertex of $Q_{i+1}(v)$. So this number is bounded by the length of $Q_{i+1}(v)$, that is $\leq n - 1$. Similarly, the total number of times that a copy of a given face of G can appear in the region $R_i(u) \setminus R_i(z_i(u))$ (corresponding to $d_i(u, z_i(u))$) is $\leq (n - 1)$.

A given face of G can appear in only one of the two gray regions of Figure 25. So a face is counted $\leq n - 1$ times in the quantity $d_i(z_i(v), v) + d_i(u, z_i(u))$. Only the faces of the region $R(C_{i+1}^j, C_{i+1}^{j+1})$ can be counted. And there is at least one face of $R(C_{i+1}^j, C_{i+1}^{j+1})$ (for example one incident to v) that is not counted. So in total $d_i(z_i(v), v) + d_i(u, z_i(u)) \leq (n - 1) \times (f_{i+1}^j - 1) < (n - 1) \times f_{i+1}^j$. \square

Clearly, the symmetric of Lemma 23, where the role of $i + 1$ and $i - 1$ are exchanged, is also true.

The bound of Lemma 23 is somehow sharp. In the example of Figure 26, the rectangle represent a toroidal map G and the universal cover is partially represented. If the map G has n vertices and f faces, then the gray region, representing the quantity $d_1(z_1(v), v) + d_1(u, z_1(u))$, has size $\frac{1}{2}(n - 1) \times (f - 1) = \Omega(n \times f)$.

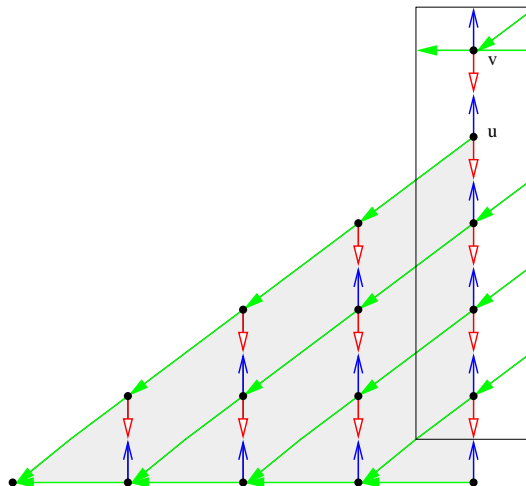


Figure 26: Example of a toroidal map where $d_1(u, z_1(u))$ has size $\Omega(n \times f)$.

Lemma 24 *Let u, v be vertices of G^∞ such that $R_i(u) \subseteq R_i(v)$, then $u_i \leq v_i$. If $R_i(u) \subsetneq R_i(v)$, then $u_i < v_i$. Moreover $v_i - u_i > (N - n)(|R(L_{i-1}(u), L_{i-1}(v))| + |R(L_{i+1}(u), L_{i+1}(v))|)$.*

Proof. We distinguish two cases depending of the fact that the Schnyder wood is of type 2.i or not.

- *Case 1: The Schnyder wood is not of Type 2.i.*

Suppose first that u and v are both in a region delimited by two consecutive lines of color $i - 1$ and two consecutive lines of color $i + 1$. Let $L_{i-1}^j, L_{i-1}^{j+1}, L_{i+1}^{j'}, L_{i+1}^{j'+1}$ be these lines such that L_{i-1}^{j+1} is on the positive side of L_{i-1}^j , $L_{i+1}^{j'+1}$ is on the positive side of $L_{i+1}^{j'}$, and $L_k^\ell \in \mathcal{L}_k^\ell$ (see Figure 27). We distinguish cases corresponding to equality or not between lines $L_{i-1}(u), L_{i-1}(v)$ and $L_{i+1}(u), L_{i+1}(v)$.

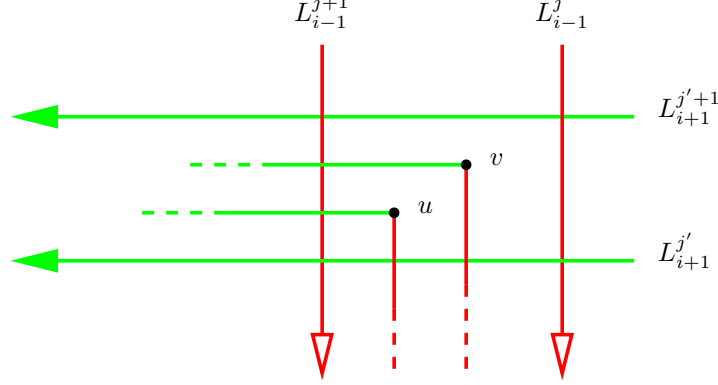


Figure 27: Position of u and v in the proof of Lemma 24

★ *Case 1.1:* $L_{i-1}(u) = L_{i-1}(v)$ and $L_{i+1}(u) = L_{i+1}(v)$. Then $v_i - u_i = d_i(v, z_i(v)) - d_i(u, z_i(u)) = d_i(v, u)$. Thus clearly, if $R_i(u) \subseteq R_i(v)$, then $u_i \leq v_i$ and if $R_i(u) \subsetneq R_i(v)$, then $u_i < v_i$. Moreover $v_i - u_i > (N - n)(|R(L_{i-1}(u), L_{i-1}(v))| + |R(L_{i+1}(u), L_{i+1}(v))|) = 0$.

★ *Case 1.2:* $L_{i-1}(u) = L_{i-1}(v)$ and $L_{i+1}(u) \neq L_{i+1}(v)$. As $u \in R_i(v)$, we have $L_{i+1}(u) = L_{i+1}^{j'}$ and $L_{i+1}(v) = L_{i+1}^{j'+1}$. Then $v_i - u_i = d_i(v, z_i(v)) - d_i(u, z_i(u)) + N(f_{i+1}(L_{i+1}(v)) - f_{i+1}(L_{i+1}(u))) = d_i(v, z_i(v)) - d_i(u, z_i(u)) + Nf_{i+1}^{j'}$. Let u' be the intersection of $P_{i+1}(u)$ with L_{i-1}^{j+1} (maybe $u = u'$). Let v' be the intersection of $P_{i+1}(v)$ with L_{i-1}^{j+1} (maybe $v = v'$). Since $L_{i+1}(u) \neq L_{i+1}(v)$, we have $u' \neq v'$. Since $u \in R_i(v)$, we have $u' \in R_i(v')$ and so $u' \in P_{i-1}(v')$. Then, by Lemma 23, $d_i(z_i(v'), v') + d_i(u', z_i(u')) < (n - 1)f_{i+1}^{j'}$. If $L_{i-1}(u) = L_{i-1}^{j+1}$ then, one can see that $d_i(v, z_i(v)) - d_i(u, z_i(u)) \geq d_i(v', z_i(v')) - d_i(u', z_i(u'))$. If $L_{i-1}(u) = L_{i-1}^j$, one can see that $d_i(v, z_i(v)) - d_i(u, z_i(u)) \geq d_i(v', z_i(v')) - d_i(u', z_i(u')) - f_{i+1}^{j'}$. So finally, $v_i - u_i = d_i(v, z_i(v)) - d_i(u, z_i(u)) + Nf_{i+1}^{j'} \geq d_i(v', z_i(v')) - d_i(u', z_i(u')) + (N - 1)f_{i+1}^{j'} > (N - n)f_{i+1}^{j'} \geq 0$.

★ *Case 1.3:* $L_{i-1}(u) \neq L_{i-1}(v)$ and $L_{i+1}(u) = L_{i+1}(v)$. Completely symmetric to the previous case.

★ *Case 1.4:* $L_{i-1}(u) \neq L_{i-1}(v)$ and $L_{i+1}(u) \neq L_{i+1}(v)$. As $u \in R_i(v)$, we have $L_{i+1}(u) = L_{i+1}^{j'}$, $L_{i+1}(v) = L_{i+1}^{j'+1}$, $L_{i-1}(u) = L_{i-1}^{j+1}$, and $L_{i-1}(v) = L_{i-1}^j$. Then $v_i - u_i = d_i(v, z_i(v)) - d_i(u, z_i(u)) + N(f_{i+1}(L_{i+1}(v)) - f_{i+1}(L_{i+1}(u))) - N(f_{i-1}(L_{i-1}(v)) - f_{i-1}(L_{i-1}(u))) = d_i(v, z_i(v)) - d_i(u, z_i(u)) + Nf_{i+1}^{j'} + Nf_{i-1}^j$. Let u' be the intersection of $P_{i+1}(u)$ with L_{i-1}^{j+1} (maybe $u = u'$). Let u'' be the intersection of $P_{i-1}(u)$ with $L_{i+1}^{j'}$ (maybe $u = u''$).

Let v' be the intersection of $P_{i+1}(v)$ with L_{i-1}^{j+1} (maybe $v = v'$). Let v'' be the intersection of $P_{i-1}(v)$ with $L_{i+1}^{j'}$ (maybe $v = v''$). Since $L_{i+1}(u) \neq L_{i+1}(v)$, we have $u' \neq v'$. Since $u \in R_i(v)$, we have $u' \in R_i(v')$ and so $u' \in P_{i-1}(v')$. Then, by Lemma 23, $d_i(z_i(v'), v') + d_i(u', z_i(u')) < (n-1)f_{i+1}^{j'}$. Symmetrically, $d_i(z_i(v''), v'') + d_i(u'', z_i(u'')) < (n-1)f_{i-1}^j$. Moreover, we have $d_i(v, z_i(v)) - d_i(u, z_i(u)) \geq d_i(v', z_i(v')) - d_i(u', z_i(u')) + d_i(v'', z_i(v'')) - d_i(u'', z_i(u'')) - f_{i+1}^{j'} - f_{i-1}^j$. So finally, $v_i - u_i = d_i(v, z_i(v)) - d_i(u, z_i(u)) + Nf_{i+1}^{j'} + Nf_{i-1}^j \geq d_i(v', z_i(v')) - d_i(u', z_i(u')) + d_i(v'', z_i(v'')) - d_i(u'', z_i(u'')) + (N-1)f_{i+1}^{j'} + (N-1)f_{i-1}^j > (N-n)f_{i+1}^{j'} + (N-n)f_{i-1}^j \geq 0$.

Suppose now that u and v do not lie in a region delimited by two consecutive lines of color $i-1$ and/or in a region delimited by two consecutive lines of color $i+1$. One can easily find distinct vertices w_0, \dots, w_r ($w_i, 1 \leq i < r$ can be chosen at intersections of monochromatic lines of colors $i-1$ and $i+1$) such that $w_0 = u$, $w_r = v$, and for $0 \leq \ell \leq r-1$, we have $w_\ell \in R_i(w_{\ell+1})$ and $w_\ell, w_{\ell+1}$ are both in a region delimited by two consecutive lines of color $i-1$ and in a region delimited by two consecutive lines of color $i+1$. Thus by the first part of the proof, $u_i \leq (w_1)_i < \dots < (w_{r-1})_i \leq v_i$. Moreover $(w_\ell)_i - (w_{\ell+1})_i \geq (N-n)(|R(L_{i-1}(w_{\ell+1}), L_{i-1}(w_\ell))| + |R(L_{i+1}(w_{\ell+1}), L_{i+1}(w_\ell))|)$. Thus $v_i - u_i \geq (N-n) \sum_{\ell} (|R(L_{i-1}(w_{\ell+1}), L_{i-1}(w_\ell))| + |R(L_{i+1}(w_{\ell+1}), L_{i+1}(w_\ell))|)$. For any a, b, c such $R_i(a) \subseteq R_i(b) \subseteq R_i(c)$, we have $|R(L_j(a), L_j(b))| + |R(L_j(b), L_j(c))| = |R(L_j(a), L_j(c))|$. Thus we obtain the result by summing the size of the regions.

• *Case 2: The Schnyder wood is of Type 2.i.*

Suppose first that u and v are both in a region delimited by two consecutive lines of color $i+1$.

Let L_{i+1}^j, L_{i+1}^{j+1} be these lines such that L_{i+1}^{j+1} is on the positive side of L_{i+1}^j , and $L_{i+1}^\ell \in \mathcal{L}_{i+1}^\ell$. We can assume that we do not have both u and v in L_{i+1}^{j+1} . We consider two cases :

★ *Case 2.1: $v \notin L_{i+1}^{j+1}$.* Then by Lemma 14, $L_{i+1}^j = L_{i+1}(u) = (L_{i-1}(u))^{-1} = L_{i+1}(v) = (L_{i-1}(v))^{-1}$. Then $v_i - u_i = d_i(v, z_i(v)) - d_i(u, z_i(u)) = d_i(v, u)$. Thus clearly, if $R_i(u) \subseteq R_i(v)$, then $u_i \leq v_i$ and if $R_i(u) \subsetneq R_i(v)$, then $u_i < v_i$. Moreover $v_i - u_i > (N-n)(|R(L_{i-1}(u), L_{i-1}(v))| + |R(L_{i+1}(u), L_{i+1}(v))|) = 0$.

★ *Case 2.2: $v \in L_{i+1}^{j+1}$.* Then $L_{i+1}^{j+1} = L_{i+1}(v) = (L_{i-1}(v))^{-1}$ and $d_i(v, z_i(v)) = 0$. By assumption $u \notin L_{i+1}^{j+1}$ and by Lemma 14, $L_{i+1}^j = L_{i+1}(u) = (L_{i-1}(u))^{-1}$. Then $v_i - u_i = d_i(v, z_i(v)) - d_i(u, z_i(u)) + N(f_{i+1}(L_{i+1}(v)) - f_{i+1}(L_{i+1}(u))) - N(f_{i-1}(L_{i-1}(v)) - f_{i-1}(L_{i-1}(u))) \geq 2Nf_{i+1}^j > (N-n)(|R(L_{i-1}(u), L_{i-1}(v))| + |R(L_{i+1}(u), L_{i+1}(v))|) > 0$.

If u and v do not lie in a region delimited by two consecutive lines of color $i+1$, then as in case 1, one can find intermediate vertices to obtain the result. \square

Lemma 25 *If two vertices u, v are adjacent, then for each color i , we have $|v_i - u_i| \leq 2Nf$.*

Proof. Since u, v are adjacent, they are both in a region delimited by two consecutive

lines of color $i - 1$ and in a region delimited by two consecutive lines of color $i + 1$. Let L_{i-1}^j, L_{i-1}^{j+1} be these two consecutive lines of color $i - 1$ and $L_{i+1}^{j'}, L_{i+1}^{j'+1}$ these two consecutive lines of color $i + 1$ with $L_k^\ell \in \mathcal{L}_k^\ell$, L_{i-1}^{j+1} is on the positive side of L_{i-1}^j and $L_{i+1}^{j'+1}$ is on the positive side of $L_{i+1}^{j'}$ (see Figure 27 when the Schnyder wood is not of Type 2.i). Let z be a vertex on the intersection of L_{i-1}^{j+1} and $L_{i+1}^{j'}$. Let z' be a vertex on the intersection of L_{i-1}^j and $L_{i+1}^{j'+1}$. Thus we have $R_i(z) \subseteq R_i(u) \subseteq R_i(z')$ and $R_i(z) \subseteq R_i(v) \subseteq R_i(z')$. So by Lemma 24, $z_i \leq u_i \leq z'_i$ and $z_i \leq v_i \leq z'_i$. So $|v_i - u_i| \leq z'_i - z_i = N(f_{i+1}(L_{i+1}^{j'+1}) - f_{i+1}(L_{i+1}^{j'})) - N(f_{i-1}(L_{i-1}^j) - f_{i-1}(L_{i-1}^{j+1})) = Nf_{i+1}^{j'} + Nf_{i-1}^j \leq 2Nf$. \square

We are now able to prove the following :

Theorem 12 *If G is a toroidal map given with a Schnyder wood, then the mapping of each vertex of G^∞ on its region vector gives a geodesic embedding of G .*

Proof. The proof follows the same steps as in the planar case (see [12]).

By Lemma 19 and 22, the mapping of G^∞ on its region vector is periodic with respect to S, S' that are not colinear. For any pair u, v of distinct vertices of G^∞ , by Lemma 13.(iii), there exists i, j with $R_i(u) \subsetneq R_i(v)$ and $R_j(v) \subsetneq R_j(u)$ thus, by Lemma 24, $u_i < v_i$ and $v_j < u_j$. So \mathcal{V}^∞ is a set of pairwise incomparable elements of \mathbb{R}^3 .

(D1) \mathcal{V}^∞ is a set of pairwise incomparable elements so the mapping between vertices of G^∞ and \mathcal{V}^∞ is a bijection.

(D2) Let $e = uv$ be an edge of G^∞ . We show that $w = u \vee v$ is on the surface $S_{\mathcal{V}^\infty}$. By definition $u \vee v$ is in $\mathcal{D}_{\mathcal{V}^\infty}$. Suppose, by contradiction that $w \notin S_{\mathcal{V}^\infty}$. Then there exist $x \in \mathcal{V}^\infty$ with $x < w$. Let x also denote the corresponding vertex of G^∞ . Edge e is in a region $R_i(x)$ for some i . So $u, v \in R_i(x)$ and thus by Lemma 13.(i) $R_i(u), R_i(v) \subseteq R_i(x)$ and by Lemma 24 $w_i = \max(u_i, v_i) \leq x_i$, a contradiction. Thus the elbow geodesic between u and v is also on the surface.

(D3) Consider a vertex $v \in \mathcal{V}$ and a color i . Let u be the extremity of the arc $e_i(v)$. We have $u \in R_{i-1}(v)$ and $u \in R_{i+1}(v)$, so by Lemma 13.(i), $R_{i-1}(u) \subseteq R_{i-1}(v)$ and $R_{i+1}(u) \subseteq R_{i+1}(v)$. Thus by Lemma 13.(iii), $R_i(v) \subsetneq R_i(u)$. So, by Lemma 24 $v_i < u_i$, $u_{i-1} \leq v_{i-1}$ and $u_{i+1} \leq v_{i+1}$. So the orthogonal arc of vertex v in direction of the basis vector e_i is part of elbow geodesic of the edge $e_i(v)$.

(D4) Suppose there exists a pair of crossing edges $e = uv$ and $e' = u'v'$ on the surface $S_{\mathcal{V}^\infty}$. The two edges e, e' cannot intersect on orthogonal arcs so they intersect on a plane orthogonal to one of the coordinate axis. Up to symmetry we may assume that we are in the situation of Figure 28 with $u_1 = u'_1$, $u_2 > u'_2$ and $v_2 < v'_2$. Between u and u' , there is a path consisting of orthogonal arcs only. With (D3), this implies that there is a bi-directed path P^* colored 0 from u to u' and colored 2 from u' to u . We have $u \in R_2(v)$, so by Lemma 13.(i), $R_2(u) \subseteq R_2(v)$. We have $u' \in R_2(u)$, so

$u' \in R_2(v)$. If $P_0(v)$ contains u' , then there is a contractible cycle containing v, u, u' in $G_1 \cup G_0^{-1} \cup G_2^{-1}$, contradicting Lemma 1, so $P_0(v)$ does not contain u' . If $P_1(v)$ contains u' , then $u' \in P_1(u) \cap P_0(u)$, contradicting Lemma 12. So $u' \in R_2^o(v)$. Thus the edge $u'v'$ implies that $v' \in R_2(v)$. So by Lemma 24, $v'_2 \leq v_2$, a contradiction. \square

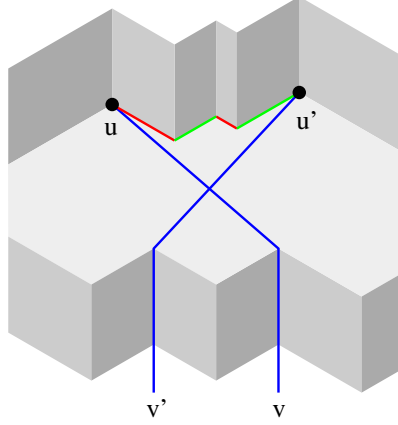


Figure 28: A pair of crossing elbow geodesic

If Conjecture 1 is true, then Theorem 12 implies that every essentially 3-connected planar maps admits a geodesic embedding. By Theorem 1, we just have the result for toroidal triangulations, thus Theorem 2 is true.

One can ask what is the “size” of the obtained geodesic embedding of Theorem 12 ? Of course this mapping is infinite so there is no real size, but as the object is periodic one can consider the smallest size of the vectors such that the mapping is periodic with respect to them. There are several such pairs of vectors, one is S, S' . Recall that $S_i = N(c_{i+1} - c_{i-1})f$ and $S'_i = N(c'_{i+1} - c'_{i-1})f$. Unfortunately the size of S, S' can be arbitrarily large. Indeed, the values of $c_{i+1} - c_{i-1}$ and $c'_{i+1} - c'_{i-1}$ are unbounded as a toroidal map can be “very twisted” in the considered flat torus representation (independently from the number of vertices or faces).

Theorem 13 *If G is a toroidal map given with a Schnyder wood, then the mapping of each vertex of G^∞ on its region vector gives a periodic mapping of G^∞ with respect to non colinear vectors Y and Y' where the size of Y and Y' is bounded by $\mathcal{O}(\gamma Nf)$. In general we have $\gamma \leq n$ and in the case where G is a simple toroidal triangulation given with a Schnyder wood obtained by Theorem 10, we have $\gamma = 1$.*

Proof. By Lemma 20, the vectors Z_{i-1}, Z_{i+1} (when the Schnyder wood is not of Type 2.i) span a subset of S, S' (it can happen that this subset is strict). Thus in the parallelogram delimited by the vectors Z_{i-1}, Z_{i+1} (that is a parallelogram by Lemma 21), there is a parallelogram with sides Y, Y' containing a copy of V . The size of the vectors Z_i is bounded by $\mathcal{O}(\gamma Nf)$ and so Y and Y' also.

In general we have $\gamma_i \leq n$ as each intersection between two monochromatic cycles of G of color $i - 1$ and $i + 1$ corresponds to a different vertex of G and thus $\gamma \leq n$. In the case of simple toroidal triangulation given with a Schnyder wood obtained by Theorem 10, we have, for each color i , $\gamma_i = 1$, and thus $\gamma = 1$. \square

We use the example of the toroidal map G of Figure 2 to illustrate the region vector method. This toroidal map has 3 vertices, 4 faces and 7 edges. There are two edges that are oriented in two directions. The Schnyder wood is of Type 1, with two 1-cycles. We choose as origin the three bold monochromatic lines of Figure 29. Then the region vectors of the vertices of G^∞ are $\{u \in \mathbb{R}^3 \mid \exists v \in \mathcal{V}, k_1, k_2 \in \mathbb{Z} \text{ such that } u = v + k_1 S + k_2 S'\}$, with $\mathcal{V} = \{(0, 0, 0), (0, 12, -11), (6, 12, -18)\}$, $S = (-12, 24, -12)$ and $S' = (12, 24, -36)$ ($N = n = 3$, $c_0 = -1$, $c_1 = 0$, $c_2 = 1$, $c'_0 = -2$, $c'_1 = 1$, $c'_2 = 0$). The points are not coplanar. They are on the two different planes of equation $x + y + z = 0$ and $x + y + z = 1$. The geodesic embedding that is obtained by mapping each vertex to its region vector is the geodesic embedding of Figure 23. The parallelogram has sides the vectors S, S' .

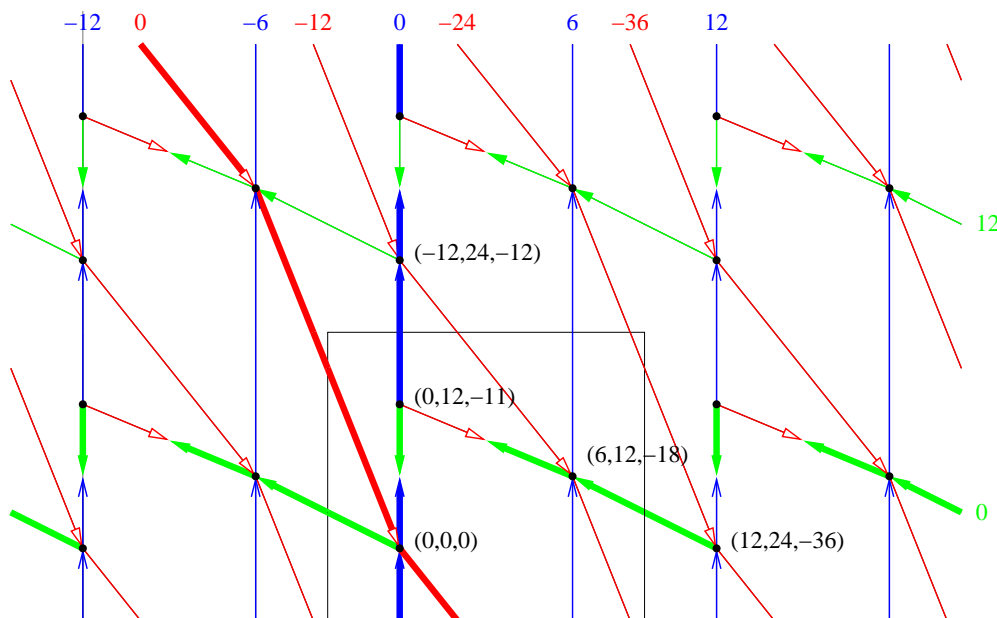


Figure 29: Coordinates of the vertices.

Like in the plane, one can give weights to faces of G . Then all their copies in G^∞ have the same weight and instead of counting the number of faces in each region one can compute the weighted sum.

Note that the geodesic embeddings of Theorem 12 are not necessarily rigid. A geodesic embedding is *rigid* if for every pair $u, v \in \mathcal{V}$ such that $u \vee v$ is in $\mathcal{S}_\mathcal{V}$, then u and v are the only elements of \mathcal{V} that are dominated by $u \vee v$ (see [11, 19]). The geodesic embedding of Figure 23 is not rigid has the bend corresponding to the loop of color 1 is dominated by three vertices of G^∞ . We do not know if it is possible to build a

rigid geodesic embedding from the Schnyder wood of a toroidal map. Maybe a technique similar to the one presented in [11] can be generalized to the torus.

It has been already mentioned that in the geodesic embeddings of Theorem 12 the points corresponding to vertices are not coplanar. The problem to build a coplanar geodesic embedding from the Schnyder wood of a toroidal map is open. In the plane, there are some examples of maps G for which it is not possible to require both rigidity and coplanarity (see [11]). Thus the same will be true in the torus for the graph G^+ .

Another question related to coplanarity is whether one can require that the points of the orthogonal surface corresponding to edges of the graph are coplanar. This property is related to contact representation by homothetic triangles (see [11]). It is known that in the plane, not all Schnyder woods are supported by such surfaces. Kratochvil's conjecture [18], recently proved [16], states that every 4-connected planar triangulation admits a contact representation by homothetic triangles. Can this be extended to the torus?

When considering non necessarily homothetic triangles, it has been proved [10] that there is a bijection between Schnyder woods of planar triangulations and contact representations by triangles. This result has been generalized to internally 3-connected planar maps in [15] by exhibiting a bijection between Schnyder woods of internally 3-connected planar maps and primal-dual contact representations by triangles (i.e. representations where both the primal and the dual are represented). It would be interesting to generalize those results to the torus.

9 Duality of orthogonal surfaces

Given an orthogonal surface generated by \mathcal{V} , let $\mathcal{F}_{\mathcal{V}}$ be the maximal points of $\mathcal{S}_{\mathcal{V}}$, i.e. the points of $\mathcal{S}_{\mathcal{V}}$ that are not dominated by any vertex of $\mathcal{S}_{\mathcal{V}}$. If $A, B \in \mathcal{F}_{\mathcal{V}}$ and $A \wedge B \in \mathcal{S}_{\mathcal{V}}$, then $\mathcal{S}_{\mathcal{V}}$ contains the union of the two line segments joining A and B to $A \wedge B$. Such arcs are called *dual elbow geodesic*. The *dual orthogonal arc* of $A \in \mathcal{F}_{\mathcal{V}}$ in the direction of the standard basis vector e_i is the intersection of the ray $A + \lambda e_i$ with $\mathcal{S}_{\mathcal{V}}$.

Given a toroidal map G , let $G^{\infty*}$ be the dual of G^{∞} . A *dual geodesic embedding* of G is a drawing of $G^{\infty*}$ on the orthogonal surface $\mathcal{S}_{\mathcal{V}^{\infty}}$, where \mathcal{V}^{∞} is a periodic mapping of G^{∞} with respect to two non colinear vectors, satisfying the following (see example of Figure 30):

- (D1*) There is a bijection between the vertices of $G^{\infty*}$ and $\mathcal{F}_{\mathcal{V}^{\infty}}$.
- (D2*) Every edge of $G^{\infty*}$ is a dual elbow geodesic.
- (D3*) Every dual orthogonal arc in $\mathcal{S}_{\mathcal{V}^{\infty}}$ is part of an edge of $G^{\infty*}$.
- (D4*) There are no crossing edges in the embedding of $G^{\infty*}$ on $\mathcal{S}_{\mathcal{V}^{\infty}}$.

Let G be a toroidal map given with a Schnyder wood. Consider the mapping of each vertex on its region vector. A face F of G^{∞} is mapped on the point $\bigvee_{v \in F} v$. We consider

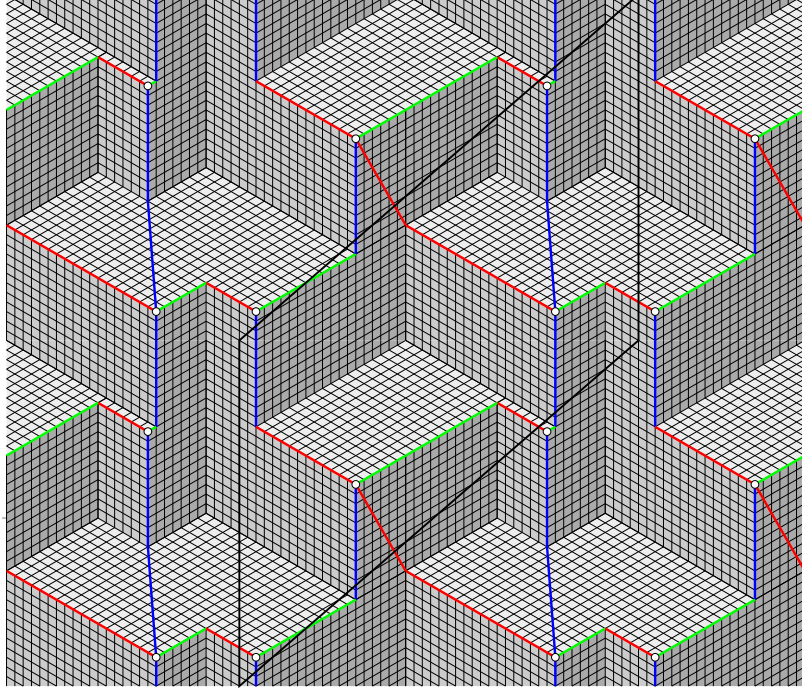


Figure 30: Dual geodesic embedding of the toroidal map of Figure 2.

the dual of the Schnyder wood of G . Let $\widetilde{G^\infty}$ be a simultaneous drawing of G^∞ and $G^{\infty*}$ such that only dual edges intersect. To avoid confusion, we note R_i the regions of the primal Schnyder wood and R_i^* the regions of the dual Schnyder wood.

Lemma 26 *For any face F of G^∞ , we have that $\bigvee_{v \in F} v$ is a maximal point of $\mathcal{S}_{\mathcal{V}^\infty}$.*

Proof. Let F be a face of G^∞ . For any vertex u of \mathcal{V}^∞ , there exists a color i , such that the face F is in the region $R_i(u)$. Thus for $v \in F$, we have $v \in R_i(u)$. By Lemma 24, we have $v_i \leq u_i$ and so $F_i \leq u_i$. So $F = \bigvee_{v \in F} v$ is a point of $\mathcal{S}_{\mathcal{V}^\infty}$.

Suppose, by contradiction, that F is not a maximal point of $\mathcal{S}_{\mathcal{V}^\infty}$. Then there is a point $\alpha \in \mathcal{S}_{\mathcal{V}^\infty}$ that dominates F and for at least one coordinate j , we have $F_j < \alpha_j$. By Lemma 16, the angles at F form, in counterclockwise order, nonempty intervals of 0's, 1's and 2's. For each color, let z^i be a vertex of F with angle i . We have F is in the region $R_i(z^i)$. So $z^{i-1} \in R_i(z^i)$ and by Lemma 13.(i), we have $R_i(z^{i-1}) \subseteq R_i(z^i)$. Since F is in $R_{i-1}(z^{i-1})$, it is not in $R_i(z^{i-1})$ and thus $R_i(z^{i-1}) \subsetneq R_i(z^i)$. Then by Lemma 24, we have $(z^{i-1})_i < (z^i)_i$ and symmetrically $(z^{i+1})_i < (z^i)_i$. So $F_{j-1} = (z^{j-1})_{j-1} > (z^j)_{j-1}$ and $F_{j+1} > (z^j)_{j+1}$. Thus α strictly dominates z^j , a contradiction to $\alpha \in \mathcal{S}_{\mathcal{V}^\infty}$. Thus F is a maximal point of $\mathcal{S}_{\mathcal{V}^\infty}$ \square

Lemma 27 *If two faces A, B are such that $R_i^*(B) \subseteq R_i^*(A)$, then $A_i \leq B_i$.*

Proof. Let $v \in B$ be a vertex whose angle at B is labeled i . We have $v \in R_i^*(B)$ and so $v \in R_i^*(A)$. In $\widetilde{G^\infty}$, the path $P_{i+1}(u)$ cannot intersect $P_{i+1}(A)$ and the path $P_{i-1}(u)$ cannot intersect $P_{i-1}(A)$. So $A \in R_i(v)$. Thus for all $u \in A$, we have $u \in R_i(v)$, so $R_i(u) \subseteq R_i(v)$, and so $u_i \leq v_i$. Then $A_i = \max_{u \in A} u_i \leq v_i \leq \max_{w \in B} w_i = B_i$. \square

Theorem 14 *If G is a toroidal map given with a Schnyder wood and each vertex of G^∞ is mapped on its region vector, then the mapping of each face of $G^{\infty*}$ on the point $\bigvee_{v \in F} v$ gives a dual geodesic embedding of G .*

Proof. (D1*) Consider a counting of elements on the orthogonal surface, where we count two copies of the same object just once (note that we are on an infinite and periodic object). We have that the sum of primal orthogonal arcs plus dual ones is exactly $3m$. There are $3n$ primal orthogonal arcs and thus there are $3m - 3n = 3f$ dual orthogonal arcs. Each maximal point of $\mathcal{S}_{\mathcal{V}^\infty}$ is incident to 3 dual orthogonal arcs and there is no dual orthogonal arc incident to two distinct maximal points. So there is at least f maximal points. Thus by Lemma 26, we have a bijection between faces of G^∞ and maximal points of $\mathcal{S}_{\mathcal{V}^\infty}$.

Let $\mathcal{V}^{\infty*}$ be the maximal points of $\mathcal{S}_{\mathcal{V}^\infty}$. Let $\mathcal{D}_{\mathcal{V}^\infty}^* = \{A \in \mathbb{R}^3 \mid \exists B \in \mathcal{V}^{\infty*} \text{ such that } A \leq B\}$. Note that the boundary of $\mathcal{D}_{\mathcal{V}^\infty}^*$ is $\mathcal{S}_{\mathcal{V}^\infty}$.

(D2*) Let $e = AB$ be an edge of $G^{\infty*}$. We show that $w = A \wedge B$ is on the surface $\mathcal{S}_{\mathcal{V}^\infty}$. By definition w is in $\mathcal{D}_{\mathcal{V}^\infty}^*$. Suppose, by contradiction that $w \notin \mathcal{S}_{\mathcal{V}^\infty}$. Then there exist C a maximal point of $\mathcal{S}_{\mathcal{V}^\infty}$ with $w < C$. By the bijection (D1*) between maximal point and vertices of $G^{\infty*}$, the point C corresponds to a vertex of $G^{\infty*}$, also denoted C . Edge e is in a region $R_i^*(C)$ for some i . So $A, B \in R_i^*(C)$ and thus, by Lemma 13.(i), $R_i^*(A) \subseteq R_i^*(C)$ and $R_i^*(B) \subseteq R_i^*(C)$. Then by Lemma 27, we have $C_i \leq \min(A_i, B_i) = w_i$, a contradiction. Thus the dual elbow geodesic between A and B is also on the surface.

(D3*) Consider a vertex A of $G^{\infty*}$ and a color i . Let B be the extremity of the arc $e_i(A)$. We have $B \in R_{i-1}^*(A)$ and $B \in R_{i+1}^*(A)$, so by Lemma 13.(i), $R_{i-1}^*(B) \subseteq R_{i-1}^*(A)$ and $R_{i+1}^*(B) \subseteq R_{i+1}^*(A)$. So by Lemma 27, $A_{i-1} \leq B_{i-1}$ and $A_{i+1} \leq B_{i+1}$. As A and B are distinct maximal point of $\mathcal{S}_{\mathcal{V}^\infty}$, they are incomparable, thus $B_i < A_i$. So the dual orthogonal arc of vertex A in direction of the basis vector e_i is part of edge $e_i(A)$.

(D4*) Suppose there exists a pair of crossing edges $e = AB$ and $e' = A'B'$ of $G^{\infty*}$ on the surface $\mathcal{S}_{\mathcal{V}^\infty}$. The two edges e, e' cannot intersect on orthogonal arcs so they intersect on a plane orthogonal to one of the coordinate axis. Up to symmetry we may assume that we are in the situation $A_1 = A'_1$, $A'_0 > A_0$ and $B'_0 < B_0$. Between A and A' , there is a path consisting of orthogonal arcs only. With (D3*), this implies that there is a bi-directed path P^* colored 2 from A to A' and colored 0 from A' to A . We have $A \in R_0(B)$, so by Lemma 13.(i), $R_0(A) \subseteq R_0(B)$. We have $A' \in R_0(A)$, so $A' \in R_0(B)$. If $P_2(B)$ contains A' , then there is a contractible cycle containing A, A', B in $G_1^* \cup G_0^{*-1} \cup G_2^{*-1}$, contradicting Lemma 1, so $P_2(B)$ does not contain A' . If $P_1(B)$ contains A' , then $A' \in P_1(A) \cap P_2(A)$, contradicting Lemma 12. So $A' \in R_0^\circ(B)$. Thus

the edge $A'B'$ implies that $B' \in R_0(B)$. So by Lemma 27, $B'_0 \geq B_0$, a contradiction. \square

Theorems 12 and 14 can be combined to obtain a simultaneous representation of a Schnyder wood and its dual on an orthogonal surface. The projection of this 3-dimensional object on the plane of equation $x + y + z = 0$ gives a representation of the primal and the dual where edges are allowed to have one bend and two dual edges have to cross on their bends (see example of Figure 31).

Theorem 15 *A toroidal triangulation admits a simultaneous flat torus representation of the primal and the dual where edges are allowed to have one bend and two dual edges have to cross on their bends. Such a representation is contained in a (triangular) grid of size $\mathcal{O}(n^3) \times \mathcal{O}(n^3)$ in general and $\mathcal{O}(n^2) \times \mathcal{O}(n^2)$ if the map is simple. Furthermore the length of the edges are bounded by $\mathcal{O}(n^2)$.*

Proof. Let G be a toroidal triangulation. By Theorems 1 (or Theorem 10 if G is simple), G admits a Schnyder wood (where monochromatic cycles of different colors intersect just once if G is simple). By Theorems 12 and 14, the mapping of each vertex of G^∞ on its region vector gives a primal and dual geodesic embedding. Thus the projection of this embedding on the plane of equation $x + y + z = 0$ gives a representation of the primal and the dual of G^∞ where edges are allowed to have one bend and two dual edges have to cross on their bends.

By Theorem 13, the obtained mapping is a periodic mapping of G^∞ with respect to non colinear vectors Y and Y' where the size of Y and Y' is bounded by $\mathcal{O}(\gamma N f)$. Thus the embedding gives a representation of G in the flat torus of sides Y, Y' .

Let $N = n$. By Euler's formula $f = 2n$, the size of the vectors Y and Y' is bounded by $\mathcal{O}(\gamma n^2)$. Thus by Theorem 13, these vectors have size bounded by $\mathcal{O}(n^3)$ in general and $\mathcal{O}(n^2)$ if the graph is simple and the Schnyder wood is obtained by Theorem 10. By Lemma 25 the length of the edges in this representation are bounded by $\mathcal{O}(n^2)$. \square

10 Straight-line representation of toroidal maps

The geodesic embedding obtained by the region vector method can be used to obtain a straight-line representation of a toroidal map (see Figure 32). For this purpose, we have to choose N bigger than previously. Note that Figure 32 is the projection of the geodesic embedding of Figure 23 obtained with the value of $N = n$. In this particular case this gives a straight-line representation but in this section we only prove that such a technique works for triangulations and for N sufficiently large. To obtain a straight-line representation of a general toroidal map, one first has to triangulate it (see Theorem 3, at the end of this section).

Let G be a toroidal triangulation given with a Schnyder wood and V^∞ the set of region vectors of vertices of G^∞ . The Schnyder wood is of Type 1 by Theorem 5. Recall

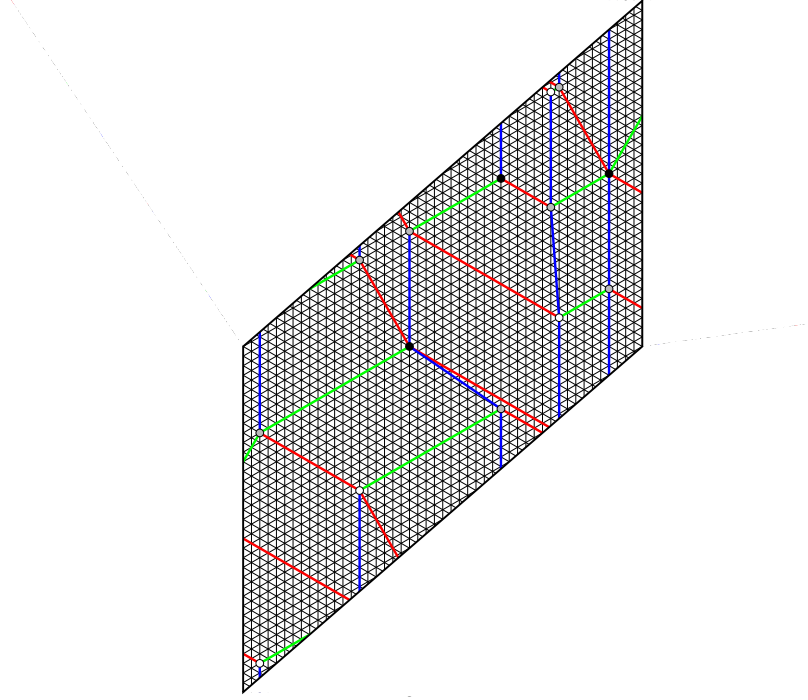


Figure 31: Simultaneous representation of the primal and the dual of the toroidal map of Figure 2 with edges having one bend.

that γ_i is the integer such that two monochromatic cycles of G of colors $i - 1$ and $i + 1$ intersect exactly γ_i times.

Lemma 28 *For any vertex v , the number of faces in the bounded region delimited by the three lines $L_i(v)$ is strictly less than $(5 \min(\gamma_i) + \max(\gamma_i))f$.*

Proof. Suppose by symmetry that $\min(\gamma_i) = \gamma_1$. Let $L_i = L_i(v)$ and $z_i = z_i(v)$. Let T be the bounded region delimited by the three monochromatic lines L_i . The boundary of T is a cycle C oriented clockwise or counterclockwise. Assume that C is oriented counterclockwise (the proof is similar if oriented clockwise). The region T is on the left sides of the lines L_i . We have $z_{i-1} \in P_i(z_{i+1})$.

We define, for $j, k \in \mathbb{N}$, monochromatic lines $L_2(j)$, $L_0(k)$ and vertices $z(j, k)$ as follows (see Figure 33). Let $L_2(1)$ be the first 2-line intersecting $L_0 \setminus \{z_1\}$ while walking from z_1 , along L_0 in the direction of L_0 . Let $L_0(1)$ be the first 0-line of color 0 intersecting $L_2 \setminus \{z_1\}$ while walking from z_1 , along L_2 in the reverse direction of L_2 . Let $z(1, 1)$ be the intersection between $L_2(1)$ and $L_0(1)$. Let $z(j, 1)$, $j \geq 0$, be the consecutive copies of $z(1, 1)$ along $L_0(1)$ such that $z(j + 1, 1)$ is after $z(j, 1)$ in the direction of $L_0(1)$. Let $L_2(j)$ be the 2-line of color 2 containing $z(j, 1)$. Note that we may have $L_2 = L_2(0)$ but in any case L_2 is between $L_2(0)$ and $L_2(1)$. Let $z(j, k)$, $k \geq 0$, be the consecutive

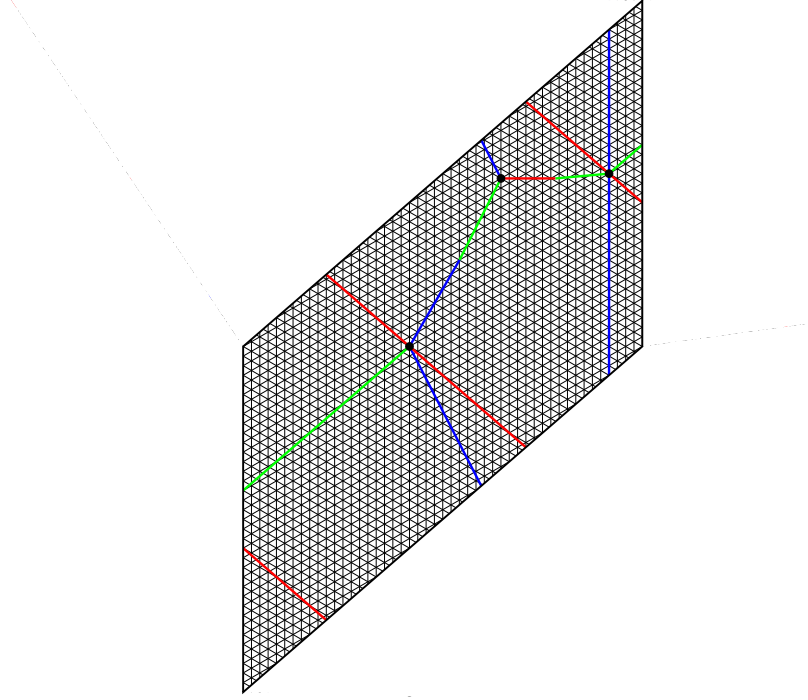


Figure 32: Straight-line representation of the graph of Figure 2 obtained by projecting the geodesic embedding of Figure 23

copies of $z(j, 1)$ along $L_2(j)$ such that $z(j, k + 1)$ is after $z(j, k)$ in the reverse direction of $L_2(j)$. Let $L_0(k)$ be the 0-line containing $z(1, k)$. Note that we may have $L_0 = L_0(0)$ but in any case L_0 is between $L_0(0)$ and $L_0(1)$. Let $S(j, k)$ be the region delimited by $L_2(j)$, $L_2(j + 1)$, $L_0(k)$, $L_0(k + 1)$. All the region $S(j, k)$ are copies of $S(0, 0)$. Each region $S(j, k)$ may contain several copies of a face of G but the number of copies of a face in $S(j, k)$ is equal to γ_1 . For $i = 0, 2$, the line L_i is on the right of $L_i(1)$. Let R be the unbounded region situated on the right of $L_0(1)$ and on the right of $L_2(1)$. As $P_0(v)$ cannot intersect $L_0(1)$ and $P_2(v)$ cannot intersect $L_2(1)$, vertex v is in R . Let $P(j, k)$ be the subpath of $L_0(k)$ between $z(j, k)$ and $z(j + 1, k)$. All the lines $L_0(k)$ are composed only of copies of $P(0, 0)$. The interior vertices of the path $P(0, 0)$ cannot contains two copies of the same vertex, otherwise there will be a vertex $z(j, k)$ between $z(0, 0)$ and $z(1, 0)$. Thus all interior vertices of a path $P(j, k)$ corresponds to distinct vertices of G .

The Schnyder wood is of Type 1, thus 1-lines are crossing 0-lines. As a line $L_0(k)$ is composed only of copies of $P(0, 0)$, any path $P(j, k)$ is crossed by a 1-line. Let L'_1 be the first 1-line crossing $P(1, 1)$ on a vertex x while walking from $z(1, 1)$ along $L_0(1)$. By (T1), line L'_1 is not intersecting $R \setminus \{z(1, 1)\}$. As $v \in R$ we have L_1 is on the left of L'_1 (maybe $L_1 = L'_1$). Thus the region T is included in the region T' delimited by L_0, L'_1, L_2 .

Let y be the vertex where L'_1 is leaving $S(1,1)$. We claim that $y \in L_2(1)$. Note that by (T1), we have $y \in L_2(1) \cup P(1,2)$. Suppose by contradiction that y is an interior vertex of $P(1,2)$. Let d_x be the length of the subpath of $P(1,1)$ between $z(1,1)$ and x . Let d_y be the length of the subpath of $P(1,2)$ between $z(1,2)$ and y . Suppose $d_y < d_x$, then there should be a distinct copy of L'_1 intersecting $P(1,1)$ between $z(1,1)$ and x on a copy of y , a contradiction to the choice of L'_1 . So $d_x \leq d_y$. Let A be the subpath of L'_1 between x and y . Let B be the subpath of $P(1,1)$ between x and the copy of y (if $d_x = d_y$, then B is just a single vertex). Consider all the copies of A and B between lines $L_2(1)$ and $L_2(2)$, they form an infinite line L situated on the right of $L_2(1)$ that prevents L'_1 from crossing $L_2(1)$, a contradiction.

By the position of x and y . We have L'_1 intersects $S(0,1)$ and $S(1,0)$. We claim that L'_1 cannot intersect both $S(0,3)$ and $S(3,0)$. Suppose by contradiction that L'_1 intersects both $S(0,3)$ and $S(3,0)$. Then L'_1 is crossing $S(0,2)$ without crossing $L_2(0)$ or $L_2(1)$. Similarly L'_1 is crossing $S(2,0)$ without crossing $L_0(0)$ or $L_0(1)$. Thus by superposing what happen in $S(0,2)$ and $S(2,0)$ in a square $S(j,k)$, we have that there are two crossing 1-lines, a contradiction. Thus L'_1 intersects at most one of $S(0,3)$ and $S(3,0)$.

Suppose that L'_1 does not intersect $S(3,0)$. Then the part of T' situated right of $L_0(2)$ is strictly included in $(S(0,0) \cup S(1,0) \cup S(2,0) \cup S(0,1) \cup S(1,1))$. Thus this part of T' contains at most $5\gamma_1 f$ faces. Now consider the part of T' situated on the left of $L_0(2)$. Let y' be the intersection of L'_1 with L_2 . Let Q be the subpath of L'_1 between y and y' . By definition of $L_2(1)$, there are no 2-lines between L_2 and $L_2(1)$. So Q cannot intersect a 2-line on one of its interior vertices. Thus Q is crossing at most γ_2 consecutive 0-lines (that are not necessarily lines of type $L_0(k)$). Let L'_0 be the $\gamma_2 + 1$ -th consecutive 0-line that is on the left of $L_0(2)$ (counting $L_0(2)$). Then the part of T' situated on the left of $L_0(2)$ is strictly included in the region delimited by $L_0(2), L'_0, L_2, L_2(1)$, and thus contains at most γ_2 copies a face of G . Thus T' contains at most $(\gamma_2 + 5\gamma_1)f$ faces.

Symmetrically if L'_1 does not intersect $S(0,3)$ we have that T' contains at most $(\gamma_0 + 5\gamma_1)f$ faces. Then in any case, T' contains at most $(\max(\gamma_0, \gamma_2) + 5\gamma_1)f$ faces and the lemma is true. \square

The bound of Lemma 28 is somehow sharp. In the example of Figure 34, the rectangle represent a toroidal triangulation G and the universal cover is partially represented. For each value of $k \geq 0$, there is a toroidal triangulation G with $n = 4(k + 1)$ vertices, where the gray region, representing the region delimited by the three monochromatic lines $L_i(v)$ contains $4 \sum_{j=1}^{2k+1} + 3(2k + 2) = \Omega(n \times f)$ faces. Figure 34 represent such a triangulation for $k = 2$.

For planar graphs the region vector method gives vertices that all lie on the same plane. This property is very helpful in proving that the position of the points on P gives straight line representations. In the torus, things are more complicated as our generalization of the region vector method does not give coplanar points. But Lemma 18 and 28 show that all the points lie in the region situated between the two planes of

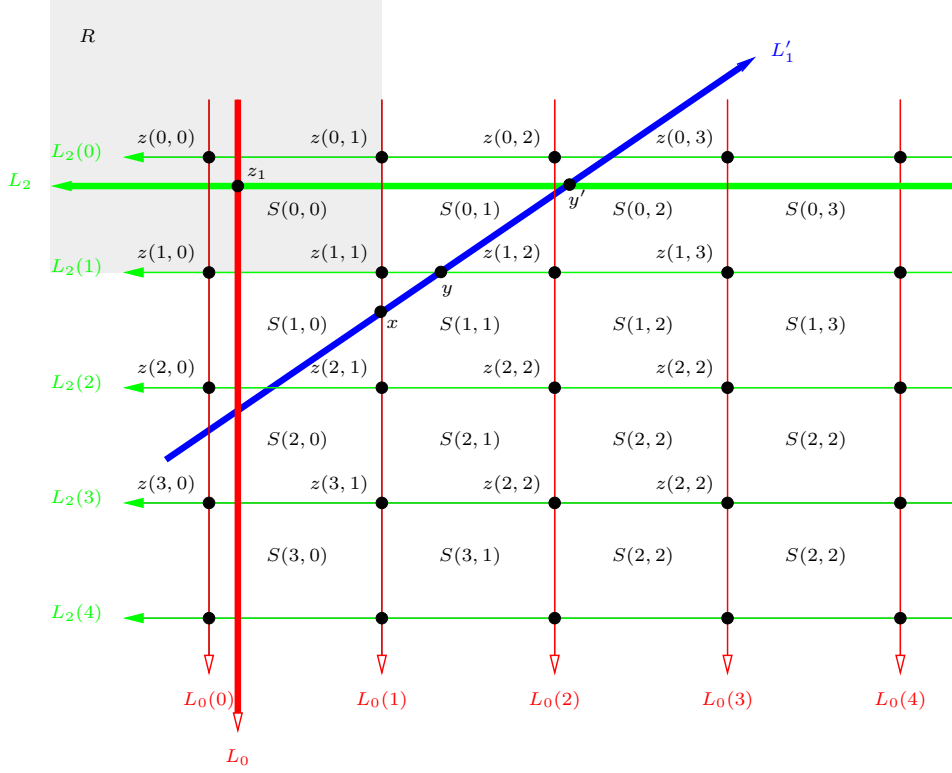


Figure 33: Notations of the proof of Lemma 28.

equation $x + y + z = 0$ and $x + y + z = t$, with $t = (5 \min(\gamma_i) + \max(\gamma_i))f$. Note that t is bounded by $6nf$ by Theorem 13 and this is independent from N . Thus from “far away” it looks like the points are coplanar and by taking N sufficiently large, non coplanar points are “far enough” from each other to enable the region vector method to give straight line representations. Let $N = t + n$.

Lemma 29 *Let u, v be two vertices such that $e_{i-1}(v) = uv$, $L_i(u) = L_i(v)$, and such that both u, v are in the region $R(L_i(u), L'_i(u))$ for $L'_i(u)$ a i -line consecutive to $L_i(u)$. Then $v_{i+1} - u_{i+1} < |R(L_i(u), L'_i(u))|$.*

Proof. Let y be the first vertex of $P_i(v)$ that is also in $P_i(u)$. Let Q_u (resp. Q_v) the part of $P_i(u)$ (resp. $P_i(v)$) between u (resp. v) and y . As $L_i(u) = L_i(v)$ and $L_{i-1}(u) = L_{i-1}(v)$, we have $v_{i+1} - u_{i+1} = d_{i+1}(v, u)$ and this is equal to the number of faces in the bounded region R delimited by the paths Q_u , Q_v , and $e_{i-1}(v)$. We have $R \subsetneq R(L_i(u), L'_i(u))$. Suppose the region R contains two copies of a given face. Then, these two copies are on different sides of a 1-line. By property (T1), it is not possible to have a 1-line entering the region R . So R contains at most one copy of each face of $R(L_i(u), L'_i(u))$. \square

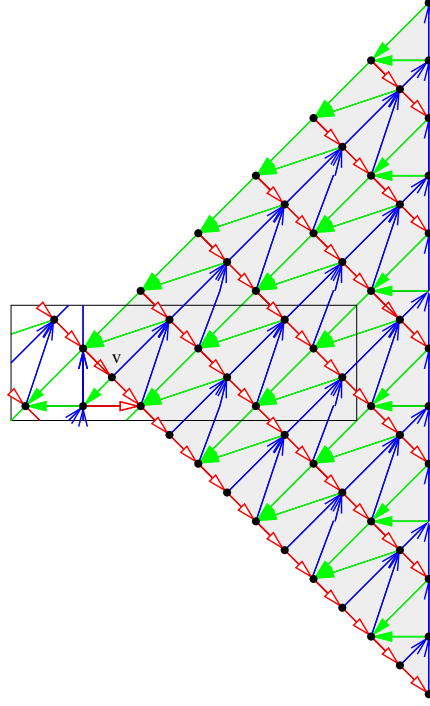


Figure 34: Example of a toroidal triangulation where the number of faces in the region delimited by the three monochromatic lines $L_i(v)$ contains $\Omega(n \times f)$ faces.

Lemma 30 *For any face F of G^∞ , incident to vertices u, v, w (given in counterclockwise order around F), the cross product $\vec{v}\vec{w} \wedge \vec{v}\vec{u}$ has strictly positive coordinates.*

Proof. Consider the angle labeling corresponding to the Schnyder wood. By Lemma 16, the angles at F are labeled in counterclockwise order 0, 1, 2. As $\vec{u}\vec{v} \wedge \vec{u}\vec{w} = \vec{v}\vec{w} \wedge \vec{v}\vec{u} = \vec{w}\vec{u} \wedge \vec{w}\vec{v}$, we may assume that u is in the angle labeled 0, vertex v in the angle labeled 1 and vertex w in the angle labeled 2. The face F is either a cycle completely directed into one direction or it has two edges oriented in one direction and one edge oriented in the other. Let

$$\vec{X} = \vec{v}\vec{w} \wedge \vec{v}\vec{u} = \begin{pmatrix} (w_1 - v_1)(u_2 - v_2) - (w_2 - v_2)(u_1 - v_1) \\ -(w_0 - v_0)(u_2 - v_2) + (w_2 - v_2)(u_0 - v_0) \\ (w_0 - v_0)(u_1 - v_1) - (w_1 - v_1)(u_0 - v_0) \end{pmatrix}$$

By symmetry, we consider the following two cases:

- *Case 1: the edges of the face F are in counterclockwise order $e_1(u)$, $e_2(v)$, $e_0(w)$ (see Figure 35.(a)).*

We have $v \in P_1(u)$, so $v \in R_0(u) \cup R_2(u)$ and $u \in R_1^\circ(v)$ (as there is no edges oriented in two direction). By Lemma 13, we have $R_0(v) \subseteq R_0(u)$ and $R_2(v) \subseteq R_2(u)$ and $R_1(u) \subsetneq R_1(v)$. In fact the first two inclusions are strict as $u \notin R_0(v) \cup R_2(v)$. So

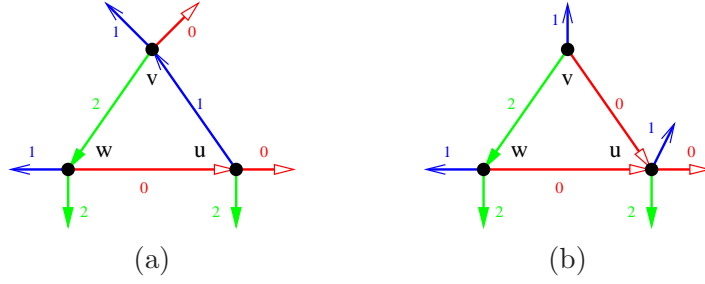


Figure 35: (a) case 1 and (b) case 2 of the proof of Lemma 30

by Lemma 24, we have $v_0 < u_0$, $v_2 < u_2$, $u_1 < v_1$. We can prove similar inequalities for the other pairs of vertices and we obtain $w_0 < v_0 < u_0$, $u_1 < w_1 < v_1$, $v_2 < u_2 < w_2$. By just studying the signs of the different terms occurring in the value of the coordinates of \vec{X} , it is clear that \vec{X} has strictly positive coordinates. (For the first coordinates, it is easier if written in the following form $X_0 = (u_1 - w_1)(v_2 - w_2) - (u_2 - w_2)(v_1 - w_1)$.)

• *Case 2: the edges of the face F are in counterclockwise order $e_0(v)$, $e_2(v)$, $e_0(w)$.* (see Figure 35.(b)).

As in the previous case, one can easily obtain the following inequalities: $w_0 < v_0 < u_0$, $u_1 < w_1 < v_1$, $u_2 < v_2 < w_2$ (the only difference with case 1 is between u_2 and v_2). Exactly like in the previous case, it is clear to see that X_0 and X_2 are strictly positive. But there is no way to reformulate X_1 to have a similar proof. Let $A = (w_2 - v_2)$, $B = (u_0 - v_0)$, $C = (v_0 - w_0)$ and $D = (v_2 - u_2)$, so $X_1 = AB - CD$ and A, B, C, D are all strictly positive.

We have $B - D = (u_0 + u_2) - (v_0 + v_2) = (v_1 - u_1) - (\sum v_i - \sum u_i)$. Since $u \in P_0(v)$, we have $L_0(u) = L_0(v)$. Suppose that $L_1(u) = L_1(v)$ and $L_2(u) = L_2(v)$, then by Lemma 18, we have $\sum u_i = \sum v_i$, and thus $B - D = v_1 - u_1 > 0$. Suppose now that $L_1(u) \neq L_1(v)$ or $L_2(u) \neq L_2(v)$. By Lemma 28, $(\sum v_i - \sum u_i) < t$. By Lemma 24, $v_1 - u_1 > (N - n)(|R(L_1(u), L_1(v))| + |R(L_2(u), L_2(v))|) \geq (N - n)$. So $B - D > N - n - t \geq 0$. Thus in both cases $B \geq D + 1$.

We have $A - C = (w_0 + w_2) - (v_0 + v_2) = (v_1 - w_1) - (\sum v_i - \sum w_i) > -(\sum v_i - \sum w_i)$. Then $X_1 = AB - CD \geq A(D + 1) - CD = A + D(A - C) > A - D(\sum v_i - \sum w_i)$. Since $w \in P_2(v)$, we have $L_2(v) = L_2(w)$. Since $u \in P_0(w) \cap P_0(v)$, we have $L_0(v) = L_0(w)$. Suppose that $L_1(v) = L_1(w)$, then by Lemma 18, we have $\sum v_i = \sum w_i$, and thus $X_1 > A > 0$. Suppose now that $L_1(v) \neq L_1(w)$. By Lemma 28, $(\sum v_i - \sum w_i) < t$. By Lemma 24, $v_1 - w_1 > (N - n)|R(L_1(v), L_1(w))|$. As $L_1(v) \neq L_1(w)$, we have $L_1(v) = L_1(u)$, and so by Lemma 29, $D = v_2 - u_2 < |R(L_1(v), L_1(w))|$. So $X_1 > (N - n - t)|R(L_1(v), L_1(w))| > 0$.

□

Consider an embedded graph H (finite or infinite) and a face F of H . Denote (f_1, f_2, \dots, f_t) the counterclockwise facial walk around F . Given a mapping of the ver-

tices of H in \mathbb{R}^2 , we say that F is *correctly oriented* if for any triplet $1 \leq i_1 < i_2 < i_3 \leq t$, the points f_{i_1} , f_{i_2} , and f_{i_3} form a counterclockwise triangle. Note that a correctly oriented face is drawn as a convex polygon.

Theorem 16 *Let G be an essentially 3-connected toroidal map given with a periodic mapping of G^∞ such that every face of G^∞ is correctly oriented. This mapping gives a straight line representation of G^∞ .*

Proof. We proceed by induction on the size of $V(G)$. Note that the theorem holds for $|V(G)| = 1$, so we assume that $|V(G)| > 1$. Given any vertex $v \in V(G)$, let $(u_0, u_1, \dots, u_{d-1})$ be the sequence of its neighbors in counterclockwise order (subscript understood modulo d). Every face being correctly oriented, for every $i \in [0, d-1]$ the oriented angle (oriented counterclockwise) $(\overrightarrow{vu_i}, \overrightarrow{vu_{i+1}}) < \pi$. Let the winding number k_v of v be the integer such that $2k_v\pi = \sum_{i \in [0, d-1]} (\overrightarrow{vu_i}, \overrightarrow{vu_{i+1}})$. It is clear that $k_v \geq 1$. Let us prove that $k_v = 1$ for every vertex v .

(4) *For any vertex v , its winding number $k_v = 1$.*

In a flat torus representation of G , we can sum up all the angles by grouping them around the vertices or around the faces.

$$\sum_{v \in V(G)} \sum_{u_i \in N(v)} (\overrightarrow{vu_i}, \overrightarrow{vu_{i+1}}) = \sum_{F \in F(G)} \sum_{f_i \in F} (\overrightarrow{f_i f_{i-1}}, \overrightarrow{f_i f_{i+1}})$$

The face being correctly oriented, they form convex polygons. Thus the angles of a face F sum at $(|F| - 2)\pi$.

$$\begin{aligned} \sum_{v \in V(G)} 2k_v\pi &= \sum_{F \in F(G)} (|F| - 2)\pi \\ \sum_{v \in V(G)} k_v &= \frac{1}{2} \sum_{F \in F(G)} |F| - f \\ \sum_{v \in V(G)} k_v &= m - f \end{aligned}$$

So by Euler's formula $\sum_{v \in V(G)} k_v = n$, and thus $k_v = 1$ for every vertex v . This proves claim (4). \diamond

Let v be a vertex of G that minimizes the number of loops whose ends are on v . Thus either v has no incident loop, or every vertex is incident to at least one loop.

Assume that v has no incident loop. Let v' be any copy of v in G^∞ and denote its neighbors $(u_0, u_1, \dots, u_{d-1})$ in clockwise order. As $k_v = 1$, the points u_0, u_1, \dots, u_{d-1} form a polygon P containing the point v' and the segments $[v', u_i]$ for any $i \in [0, d-1]$. It is well known that any polygon, admits a partition into triangles by adding some of the chords. Let us call O the maximum outerplanar graph with outer boundary

$(u_0, u_1, \dots, u_{d-1})$, obtained by this “triangulation” of P . Let us now consider the toroidal map $G' = (G \setminus \{v\}) \cup O$ and its periodic embedding obtained from the mapping of G^∞ by removing the copies of v . It is easy to see that in this embedding every face of G' is correctly oriented (including the inner faces of O , or the faces of G that have been shortened by an edge $u_i u_{i+1}$). Thus by induction hypothesis, the mapping gives a straight line representation of G'^∞ . It is also a straight line representation of G^∞ minus the copies of v where the interior of each copy of the polygons P are pairwise disjoint and do not intersect any vertex or edge. Thus one can add the copies of v on their initial positions and add the edges with their neighbors without intersecting any edge. The obtained drawing is thus a straight line representation of G^∞ .

Assume now that every vertex is incident to at least one loop. Since these loops are non-contractible and do not cross each other, they form homothetic cycles. Thus G is as depicted in Figure 36, where the dotted segments stand for edges that may be in G but not necessarily. Since the mapping is periodic the edges corresponding to loops of G form several parallel lines, cutting the plane into infinite strips. Since for any $i \in [n]$, $k_{v_i} = 1$, a line of copies of v_i divides the plane, in such a way that their neighbors which are copies of v_{i-1} and their neighbors which are copies of v_{i+1} are in distinct half-planes. Thus adjacent copies of v_i and v_{i+1} are on two lines bounding a strip. Then one can see that the edges between copies of v_i and v_{i+1} are contained in this strip without intersecting each other. Thus the obtained mapping of G^∞ is a flat torus straight line representation. \square

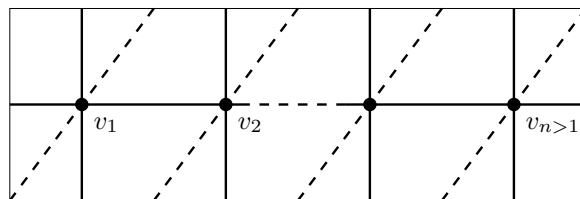


Figure 36: The graph G if every vertex is incident to a loop.

A plane is *positive* if it has equation $\alpha x + \beta y + \gamma z = 0$ with $\alpha, \beta, \gamma \geq 0$.

Theorem 17 *If G is a toroidal triangulation given with a Schnyder wood, and V^∞ the set of region vectors of vertices of G^∞ . Then the projection of V^∞ on a positive plane gives a straight-line representation of G^∞ .*

Proof. Let $\alpha, \beta, \gamma \geq 0$ and consider the projection of V^∞ on the plane P of equation $\alpha x + \beta y + \gamma z = 0$. A normal vector of the plane is given by the vector $\vec{n} = (\alpha, \beta, \gamma)$. Consider a face F of G^∞ . Suppose that F is incident to vertices u, v, w (given in counterclockwise order around F). By Lemma 30, $(\vec{uv} \wedge \vec{vw}) \cdot \vec{n}$ is positive. Thus the projection of the face F on P is correctly oriented. So by Theorem 16, the projection of V^∞ on P gives a straight-line representation of G^∞ . \square

We can now combine the different results to prove Theorem 3.

Proof of Theorem 3. Let G be any toroidal map. A toroidal map can be triangulated (by keeping it simple if it was simple) by adding a linear number of vertices and edges. Let G' be a toroidal triangulation such that G is a subgraph of G' (and G' is simple if and only if G is simple). By Theorems 1 (or Theorem 10 if G' is simple), G' admits a Schnyder wood (where monochromatic cycles of different colors intersect just once if G' is simple). By Theorem 17, the projection of the set of region vectors of vertices of G'^∞ on a positive plane gives a straight-line representation of G'^∞ .

By Theorem 13, the obtained mapping is a periodic mapping of G^∞ with respect to non colinear vectors Y and Y' where the size of Y and Y' is bounded by $\mathcal{O}(\gamma N f)$. Thus the embedding gives a straight line representation of G in the flat torus of side Y, Y' .

We have $N \leq 6\gamma f + n$, so the size of the vectors Y and Y' is bounded by $\mathcal{O}(\gamma^2 f^2)$. Thus by Theorem 13, these vectors have size bounded by $\mathcal{O}(n^4)$ in general and $\mathcal{O}(n^2)$ if the graph is simple and the Schnyder wood is obtained by Theorem 10. By Lemma 25 the length of the edges in this representation are bounded by $\mathcal{O}(n^3)$ in general and by $\mathcal{O}(n^2)$ if the graph is simple. \square

The grid where the straight line flat torus representation is obtained in the proof of Theorem 3 is either the triangular grid (if the projection is done on the plane of equation $x + y + z = 0$) or the square grid (if the projection is done on one of the plane of equation $x = 0$, $y = 0$ or $z = 0$).

The proof of Theorem 3 gives in fact a polynomial algorithm to obtain straight line representation of any toroidal maps. First triangulate your map, then find a Schnyder wood as explained in Section 7, then compute the region vector of one copy of each vertex of G in G^∞ , compute vectors Z_0, Z_1 , use these vectors to find the vectors Y, Y' , finally project these vertices on the desired plane. If the map is simple and one wants to obtain better bounds on the size of the grids, then one has to triangulate it by keeping it simple and use Theorem 10. The proof of Theorem 10 also leads to a polynomial algorithm. It uses Theorem 9 [13] which in turn uses results from Robertson and Seymour [23] on disjoint paths problems.

It would be nice to extend Theorem 17 to be true for any toroidal map given with a Schnyder wood as then it may be used to obtain convex straight line representation for essentially 3-connected toroidal maps if Conjecture 1 appears to be true.

11 Conclusion

We have proposed a generalization of Schnyder woods to toroidal maps with application to graph drawing. Along these lines, several questions were raised. We recall some briefly:

- Does the set of Schnyder woods of a given toroidal map has a kind of lattice structure ?

- Does any essentially 3-connected toroidal map admits a Schnyder wood ?
- Does any simple toroidal triangulation admits a Schnyder wood where the set of edges of each color induces a connected subgraph ?
- Is it possible to use Schnyder woods to embed toroidal maps on rigid or coplanar orthogonal surfaces ?
- Which toroidal maps admits (primal-dual) contact representation by (homothetic) triangles in a flat torus ?
- Can geodesic embeddings be used to obtain convex straight line representation for essentially 3-connected toroidal maps ?

Planar Schnyder woods appear to have many applications in various areas like enumeration [1], compact coding [22], representation by geometric objects [10, 15], graph spanners [2], graph drawing [7, 17], etc. In this paper we have shown how our definition of Schnyder wood can be used for graph drawing purpose, it would also be interesting to see if it can be used in other computer science domains.

The guideline of Castelli Aleardi et al. [3] to generalize Schnyder wood to higher genus was to preserve the tree structure of planar Schnyder woods and to use this structure for efficient encoding. For that purpose they introduce several special rules (even in the case of genus 1). Our main guideline while working on this paper was that the surface of genus 1, the torus, seems to be the perfect surface to define Schnyder woods. Euler's formula gives exactly $m = 3n$ for toroidal triangulations, and this is a necessary condition for every vertex satisfying the local Schnyder property. Thus a simple and symmetric object can be defined by relaxing the tree constraint. For genus 0, the plane, there are not enough edges in planar triangulations to have outdegree three for every vertex. Thus there is a need of a special rule like the half edges and double orientation for the outer face. For higher genus (the double torus, ...) there are too many edges in triangulations, so there is also a need for special rules. Another open problem is to find what would be the natural generalization of our definition of toroidal Schnyder woods to higher genus.

Acknowledgments

The authors thank Nicolas Bonichon, Luca Castelli Aleardi and Eric Fusy for fruitful discussions about this work. They also thank a student Chloe Desdouts for developping a software to visualize orthogonal surfaces.

References

- [1] N. Bonichon, A bijection between realizers of maximal plane graphs and pairs of non-crossing dyck paths, *Discrete Mathematics* 298 (2005) 104-114.

- [2] N. Bonichon, C. Gavaille, N. Hanusse, D. Ilcinkas, Connections between Theta-Graphs, Delaunay Triangulations, and Orthogonal Surfaces. WG10 (2010).
- [3] L. Castelli Aleardi, E. Fusy, T. Lewiner, Schnyder woods for higher genus triangulated surfaces, with applications to encoding, *Discrete and Computational Geometry* 42 (2009) 489-516.
- [4] E. Chambers, D. Eppstein, M. Goodrich, M. Löffler, Drawing graphs in the plane with a prescribed outer face and polynomial area, *Lecture Notes in Computer Science* 6502 (2011) 129-140.
- [5] C. Duncan, M. Goodrich, S. Kobourov, Planar drawings of higher-genus graphs, *Journal of Graph Algorithms and Applications* 15 (2011) 13-32.
- [6] B. Dushnik, E.W. Miller, Partially ordered sets, *American Journal of Mathematics* 63 (1941) 600-610.
- [7] S. Felsner, Convex Drawings of Planar Graphs and the Order Dimension of 3-Polytopes, *Order* 18 (2001) 19-37.
- [8] S. Felsner, Geodesic Embeddings and Planar Graphs, *Order* 20 (2003) 135-150.
- [9] S. Felsner, Lattice structures from planar graphs, *Electron. J. Combin.* 11 (2004).
- [10] H. de Fraysseix, P. Ossona de Mendez, P. Rosenstiehl, On Triangle Contact Graphs, *Combinatorics, Probability and Computing* 3 (1994) 233-246.
- [11] S. Felsner, F. Zickfeld, Schnyder Woods and Orthogonal Surfaces, *Discrete Comput Geom* 40 (2008) 103-126.
- [12] S. Felsner, *Geometric Graphs and Arrangements*, Vieweg, 2004.
- [13] G. Fijavz, personal communication (2011).
- [14] H. de Fraysseix, P. Ossona de Mendez, On topological aspects of orientations, *Discrete Mathematics* 229 (2001) 57-72.
- [15] D. Gonçalves, B. Lévêque, A. Pinlou, Triangle contact representations and duality, to appear in *Discrete and Computational Geometry*.
- [16] D. Gonçalves, B. Lévêque, A. Pinlou, Homothetic triangle representations of planar graphs, manuscript, 2011.
- [17] G. Kant, Drawing planar graphs using the canonical ordering, *Algorithmica* 16 (1996) 4-32.
- [18] J. Kratochvíl, Bertinoro Workshop on Graph Drawing, 2007.
- [19] E. Miller, Planar graphs as minimal resolutions of trivariate monomial ideals, *Documenta Mathematica* 7 (2002) 43-90.

- [20] B. Mohar, Straight-line representations of maps on the torus and other flat surfaces, *Discrete Mathematics* 155 (1996) 173-181.
- [21] B. Mohar, P. Rosenstiehl, Tessellation and visibility representations of maps on the torus, *Discrete Comput. Geom.* 19 (1998) 249-263.
- [22] D. Poulalhon, G. Schaeffer, Optimal coding and sampling of triangulations, *Algorithmica* 46 (2006) 505-527.
- [23] N. Robertson, P.D Seymour, Graph minors. VI. Disjoint paths across a disc, *Journal of Combinatorial Theory B*, 41 (1986) 115-138.
- [24] W. Schnyder, Planar graphs and poset dimension, *Order* 5 (1989) 323-343.